THE GLOBAL OPTIMIZATION OF PHASE-INCOHERENT SIGNALS

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ABSTRACT

The problem of global optimization of M phase-incoherent signals in N complex dimensions is formulated. Then, by using the geometric approach of Landau and Slepian, conditions for optimality are established for $N = 2$ and the optimal signal sets are determined for $M = 2, 3, 4, 6$, and 12.

The method is the following: The signals are assumed to be equally probable and to have equal energy, and thus are represented by points $\bar{s}_i$, $i = 1, 2, \cdots, M$, on the unit sphere S_1 in C^N. If W_{ik} is the halfspace determined by $\bar{s}_i$ and $\bar{s}_k$ and containing $\bar{s}_i$, i.e. $W_{ik} = \{\bar{r} \in C^N : |\langle \bar{r}, \bar{s}_i \rangle| \geq |\langle \bar{r}, \bar{s}_k \rangle|\}$, then the $\mathcal{R}_i = \bigcap_{k \neq i} W_{ik}$, $i = 1, 2, \cdots, M$, the maximum likelihood decision regions, partition S_1. For additive complex Gaussian noise $\bar{n}$ and a received signal $\bar{r} = \bar{s}_i e^{j\theta} + \bar{n}$, where θ is uniformly distributed over $[0, 2\pi]$, the probability of correct decoding is $P_C = \frac{1}{\pi^N} \int_0^\infty r^{2N-1} e^{-(r^2+1)} U(r) dr$,

where $U(r) = \frac{1}{M} \sum_{i=1}^{M} \int_{\mathcal{R}_i \cap S_1} I_0(2r |\langle \bar{s}, \bar{s}_i \rangle|) d\sigma(\bar{s})$, and $r = \|\bar{r}\|$.

For $N = 2$, it is proved that $U(r) \leq \int_{C_\alpha} I_0(2r |\langle \bar{s}, \bar{s}_i \rangle|) d\sigma(\bar{s}) - \frac{2K}{M} \cdot h\left(\frac{1}{2K} [M\sigma(C_\alpha) - \sigma(S_1)]\right)$, where $C_\alpha = \{\bar{s} \in S_1 : |\langle \bar{s}, \bar{s}_i \rangle| \geq \alpha\}$, K is the total number of boundaries of the net on S_1 determined by the decision regions, and h is the strictly increasing strictly convex function of $\sigma(C_\alpha \cap W)$, (where W is a halfspace not containing $\bar{s}_i$),

given by $h = \int_{C_\alpha \cap W} I_0(2r|\langle \overline{s}, \overline{s}_i \rangle|)d\sigma(\overline{s})$. Conditions for equality are established and these give rise to the globally optimal signal sets for $M = 2, 3, 4, 6$, and 12.

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INTRODUCTION

The problem of optimal (minimizing the probability of error) signal selection for transmission of messages over phase-coherent and phase-incoherent channels has been a subject of many investigations. Under the assumption of additive white Gaussian noise, equal energy, and equiprobable signal sets, Balakrishnan [1] showed in 1961 that with no bandwidth constraint the regular simplex is globally optimal for small and large signal-to-noise ratios for the phase-coherent channel. Landau and Slepian [2] established in 1966 that, in fact, the regular simplex code is globally optimal for the phase-coherent channel independent of the signal-to-noise ratio and for a larger class of probability density functions.

Also in 1966, using the approach of Balakrishnan, Scholtz and Weber [3] proved that the orthogonal signal set is locally optimal for the phase-incoherent channel under no bandwidth constraint. For M phase-incoherent signals in M-1 dimensions, i.e., a bandwidth constraint, the signals with $|\langle \bar{s}_i, \bar{s}_j \rangle| = \frac{1}{M-1}$ were established as locally optimal by Weber [4] in 1967.

Using the geometric approach of Landau and Slepian, we formulate a condition for global optimality of M equi-probable phase-incoherent signals in N complex dimensions. In the geometric approach, the length of the signal vectors is proportional to energy; and the dimensionality of the space is analogous to bandwidth [8]. For the set of probability densities which are monotone decreasing away from the signal vectors (of which the Gaussian is a member), we prove the validity of these conditions for N = 2 along with some related necessary conditions. We then perform a transformation which maps the unit sphere in C^2 onto the unit sphere in three-dimensional Euclidean space. With this transformation, we are able to use Euler's formula to show that the global solutions obtainable by this method are M = 2, 3, 4, 6, and 12; and these have respectively 1, 2, 3, 4, and 5 hyperplanes forming the boundary of their decision regions. We then obtain the globally optimal signal sets for these M's.

In particular, we demonstrate that the signal sets which are <u>globally</u> optimal in two complex dimensions are, in fact, the above-mentioned signal sets for $M = 2$ and $M = 3$ (i.e., the orthogonal signal set $\langle \bar{s}_i, \bar{s}_j \rangle = 0$ for two signals and $|\langle \bar{s}_i, \bar{s}_0 \rangle| = \frac{1}{2}$ for three signals). For four signals, the globally optimal signal set has $|\langle \bar{s}_i, \bar{s}_j \rangle| = \frac{1}{\sqrt{3}}$. For six and twelve signals, the inner product between the signal vector and the ones determining the decision region are given by $|\langle \bar{s}_i, \bar{s}_j \rangle| = \frac{1}{\sqrt{2}}$ for six signals and $|\langle \bar{s}_i, \bar{s}_j \rangle| = \sqrt{\frac{2}{5 - \sqrt{5}}}$ for twelve signals.

CHAPTER I

THE OPTIMUM RECEIVER AND THE EQUIVALENT VECTOR CHANNEL

In this Chapter we derive the optimum receiver for the transmission of messages over a phase-incoherent channel. In this derivation, it is assumed that the noise is additive white Gaussian and that the messages are all equi-probable and have equal energy.

Let $\{A_i x_i(t),\ i = 1, 2, \cdots, M\}$ be the set of real messages to be transmitted where $A_i x_i(t)$ is defined on $0 \le t \le T$ and has energy $\frac{A_i^2}{2}$. Let $y_i(t)$ be the Hilbert transform of $x_i(t)$, i.e., $y_i(t) = \frac{1}{\pi t} \circledast x_i(t)$.

Let $s_i(t)$ be the complex message defined by $s_i(t) = x_i(t) + j y_i(t)$ which has spectrum

$$S_i(f) = \begin{cases} 2X_i(f) & f \ge 0 \\ 0 & f < 0 \end{cases} \tag{1}$$

and having unit energy.

Let $n(t)$ be complex white noise with zero mean and power spectrum

$$S_n(f) = \begin{cases} 2N_0 & f \ge 0 \\ 0 & f < 0 \end{cases} \tag{2}$$

Next we assume that the received signal is of the form

$$r(t) = A_i s_i(t)\, e^{j\theta} + n(t) \tag{3}$$

where θ represents the phase of the r-f carrier and has a uniform probability density defined on the interval $0 \leq \theta \leq 2\pi$, i.e.,

$$p(\theta) = \begin{cases} \frac{1}{2\pi} & \theta \in [0, 2\pi] \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

We now let $\{\varphi_i(t)\}_{i=1,2,\cdots,N}$ be a set of complex orthonormal basis functions for the linear space spanned by the $\{s_i(t)\}_{i=1,2,\cdots,M}$. Then we define the k^{th} component of the r vector as

$$r^k = \int_0^T r(t)\, \varphi_k^*(t)\, dt \tag{5}$$

and similarly

$$s_i^k = \int_0^T s_i(t)\, \varphi_k^*(t)\, dt \tag{6}$$

and $$n^k = \int_0^T n(t)\, \varphi_k^*(t)\, dt\ . \tag{7}$$

This yields for the k^{th} component the equation $r^k = A_i s_i^k e^{j\theta} + n^k$ and hence we obtain the vector equation

$$\overline{r} = A_i \overline{s}_i e^{j\theta} + \overline{n}\ . \tag{8}$$

The minimum probability of error receiver is then to select the i^{th} messages as being transmitted when

$$p(\overline{r} | A_i \overline{s}_i)\, p(\overline{s}_i) = \max_k p(\overline{r} | A_k \overline{s}_k)\, p(\overline{s}_k) \tag{9}$$

Assuming complex Gaussian distributed noise with zero mean and variance $2N_0$, we obtain

$$p(\bar{r}|A_i\bar{s}_i,\theta) = \frac{1}{(2\pi N_0)^N} e^{-\frac{1}{2N_0}\|\bar{r}-A_i\bar{s}_i e^{j\theta}\|^2} . \tag{10}$$

Then

$$p(\bar{r}|A_i\bar{s}_i) = \int_0^{2\pi} p(\bar{r}|A_i\bar{s}_i,\theta)\, p(\theta)\, d\theta \tag{11}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(2\pi N_0)^N} e^{-\frac{1}{2N_0}\|\bar{r}-A_i\bar{s}_i e^{j\theta}\|^2} d\theta$$

$$= \frac{1}{(2\pi N_0)^N} e^{-\frac{\|\bar{r}\|^2 + A_i^2}{2N_0}} \left[\frac{1}{2\pi}\int_0^{2\pi} e^{Re\left[\frac{\langle \bar{r}, A_i\bar{s}_i e^{j\theta}\rangle}{N_0}\right]} d\theta\right] \tag{12}$$

But $\frac{1}{2\pi}\int_0^{2\pi} e^{Re\left[\frac{\langle \bar{r}, A_i\bar{s}_i e^{j\theta}\rangle}{N_0}\right]} d\theta$ is known to be $I_0\left[\frac{A_i|\langle \bar{r},\bar{s}_i\rangle|}{N_0}\right]$,

where I_0 is the modified Bessel function of the first kind. The minimum probability of error decision rule is to select the i^{th} message as being transmitted such that $\bar{s}_i$ maximizes

$$I_0\left(\frac{A_i|\langle\bar{r},\bar{s}_i\rangle|}{N_0}\right) e^{-\frac{A_i^2}{2N_0}} p(\bar{s}_i) \tag{13}$$

We now assume that the messages are all equi-probable and have probability $p(\bar{s}_i) = \frac{1}{M}$ and equal energy $A_i^2 = A^2$. Then the optimum decision rule reduces to selecting $\bar{s}_i$ to maximize

$$I_0\left(\frac{A|\langle\bar{r},\bar{s}_i\rangle|}{N_0}\right)$$

Now

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2^n n!)^2} \tag{14}$$

which is a monotone increasing function of $|x|$. Therefore the optimum decision rule is to select $\bar{s}_i$ such that

$$|\langle\bar{r},\bar{s}_i\rangle| = \max_k |\langle\bar{r},\bar{s}_k\rangle| \tag{15}$$

For convenience in later sections, we let $N_0 = \frac{1}{2}$ then A^2 represents the signal-to-noise power ratio of the complex signal. Also, using this notation, we see that all signal vectors $\{\bar{s}_i\}_{i=1,2,\cdots,M}$ have unit energy and may be considered as points on the unit sphere in C^N.

CHAPTER II

FORMULATION OF THE PROBLEM

In this Chapter, we use the equivalent vector channel presented in Chapter I. In this formulation, the set of signal vectors $\{\bar{s}_i\}_{i=1,2,\cdots,M}$ all have unit energy (i.e., $\|\bar{s}_i\| = 1$); and hence we represent them as points on the unit sphere in C^N. The dimensionality of the complex space is proportional to the bandwidth of the communication system.[8] The received vector is assumed to be of the form $\bar{r} = \bar{s}_i e^{j\theta} + \bar{n}$ where $s_i(t) = x_i(t) + jy_i(t)$ and $x_i(t)$ is the real message transmitted where $y_i(t)$ is the Hilbert transform of $x_i(t)$. The noise $\bar{n}$ is assumed to be complex additive Gaussian noise with zero mean and variance one. θ represents the unknown phase of the r-f carrier and is assumed to be uniformly distributed on $[0, 2\pi]$.

The probability density of receiving a vector $\bar{r}$, given that $x_i(t)$ $(x_i(t) \rightarrow s_i(t))$ was transmitted, is then seen to be (Chapter I, Eq.(12))

$$P(\bar{r}|\bar{s}_i) = \frac{1}{\pi^N} e^{-(r^2+1)} \frac{1}{2\pi}\int_0^{2\pi} e^{2|\langle\bar{r},\bar{s}_i\rangle|\cos\theta}\, d\theta$$

$$= \frac{1}{\pi^N} e^{-(r^2+1)} I_0\left(2r\left|\left\langle\frac{\bar{r}}{r}, \bar{s}_i\right\rangle\right|\right) \tag{16}$$

where $r = \|\bar{r}\|$. Thus, we can write

$$P(\bar{r}|\bar{s}_i) = \frac{1}{\pi^N} e^{-(r^2+1)} P_r\left(\left|\left\langle\frac{\bar{r}}{r}, \bar{s}_i\right\rangle\right|\right) \tag{17}$$

where

$$P_r\left(\left|\left\langle \frac{\bar{r}}{r}, \bar{s}_i \right\rangle\right|\right) = I_0\left(2r\left|\left\langle \frac{\bar{r}}{r}, \bar{s}_i \right\rangle\right|\right) \tag{18}$$

is for each fixed $r > 0$ a strictly increasing function on $[0,1]$.

We may partition C^N into M decision regions

$$\mathcal{R}_i = \{\bar{r}\in C^N : |\langle \bar{r}, \bar{s}_i\rangle| \geq |\langle \bar{r}, \bar{s}_k\rangle| \,\forall\, k \neq i\}, \tag{19}$$

and each $\mathcal{R}_i$ region contains $\bar{s}_i e^{j\theta}$ for all $0 \leq \theta \leq 2\pi$, $i = 1, 2, \cdots, M$. We may now write the probability of no decoding error (Q) as

$$Q = \sum_{i=1}^{M} p(\bar{s}_i) \int_{\mathcal{R}_i} p(\bar{r}|\bar{s}_i)\, dm(\bar{r})$$

$$= \sum_{i=1}^{M} p(\bar{s}_i) \int_{\mathcal{R}_i} \frac{1}{\pi^N} e^{-(r^2+1)} P_r\left(\left|\left\langle \frac{\bar{r}}{r}, \bar{s}_i \right\rangle\right|\right) dm(\bar{r})$$

$$= \frac{1}{\pi^N} \int_0^\infty r^{2N-1} e^{-(r^2+1)}\, U(r)\, dr \tag{20}$$

where

$$U(r) = \sum_{i=1}^{M} p(\bar{s}_i) \int_{\frac{\mathcal{R}_i \cap S_r}{r}} P_r(|\langle \bar{s}, \bar{s}_i\rangle|)\, d\sigma(\bar{s}) \tag{21}$$

and $S_{r_0} = \{\bar{r}\in C^N : \|\bar{r}\| = r_0\}$.

Assuming equally probable signals, Eq.(21) can be rewritten as

$$U(r) = \frac{1}{M} \sum_{i=1}^{M} \int_{\frac{\mathcal{R}_i \cap S_r}{r}} P_r(|\langle \bar{s}, \bar{s}_i \rangle|)\, d\sigma(\bar{s}) \tag{22}$$

and, clearly, Q is maximized if $U(r)$ is maximized for each $r > 0$. Now, we let

$$R_i = \mathcal{R}_i \cap S_1 = \frac{\mathcal{R}_i \cap S_r}{r} \quad . \tag{23}$$

Note, also, if we let W_{ik} be a half-space determined by $\bar{s}_i$ and $\bar{s}_k$ containing $\bar{s}_i$ and defined by

$$W_{ik} = \{\bar{r} \in C^N : |\langle \bar{r}, \bar{s}_i \rangle| \geq |\langle \bar{r}, \bar{s}_k \rangle|\} \quad , \tag{24}$$

then

$$\mathcal{R}_i = \bigcap_{k \neq i} W_{ik} \quad . \tag{25}$$

Consequently, our problem is to find a condition on the location of the points $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_M$ on the unit sphere in C^N such that $Q = \frac{1}{\pi^N} \int_0^\infty r^{2N-1} e^{-(r^2+1)} U(r)\, dr$ is maximized. Where $U(r) = \frac{1}{M} \sum_{i=1}^{M} \int_{R_i} P_r(|\langle \bar{s}, \bar{s}_i \rangle|)\, d\sigma(\bar{s})$, P_r is an increasing function of $[0,1]$, $R_i = \mathcal{R}_i \cap S_1$; and the decision regions R_i are the intersection of a finite number of half-spaces of C^N determined by points on S_1.

CHAPTER III

THE METHOD OF LANDAU AND SLEPIAN

In this Chapter, we present the method of Landau and Slepian modified for the phase-incoherent optimization problem.

For $0 < \alpha < 1$, we define the "cap" of S_1 centered at $\bar{s}_i$ and of size α to be

$$C_{i,\alpha} = \{\bar{s} \in S_1 : |\langle \bar{s}, \bar{s}_i \rangle| \geq \alpha\} \ . \tag{27}$$

We let $\sigma(C_\alpha)$ denote the common value of $\sigma(C_{i,\alpha})$, $i = 1, 2, \cdots, M$ and further restrict $\sigma(C_\alpha)$ such that

$$\frac{1}{M}\,\sigma(S_1) \leq \sigma(C_\alpha) \leq \frac{1}{2}\,\sigma(S_1) \quad . \tag{28}$$

If W is a half space which does not contain $\bar{s}_i$, let

$$h = \int_{C_{i,\alpha} \cap W} P_r(|\langle \bar{s}, \bar{s}_i \rangle|)\, d\sigma(\bar{s}) \ . \tag{29}$$

The method of Landau and Slepian is based on proving the following properties of h which we shall prove in Chapter IV for $N = 2$:

(A) h is a function only of $\sigma(C_{i,\alpha} \cap W)$ for fixed α and, in fact, is a strictly increasing strictly convex function.

(B) If V is the intersection of a finite number of half-spaces, at least one of which does not contain $\bar{s}_i$, then

$$\int_{C_{i,\alpha} \cap V} P_r(|\langle \bar{s}, \bar{s}_i \rangle|)\, d\sigma(\bar{s}) \geq h(\sigma(C_{i,\alpha} \cap V)) \tag{30}$$

with equality if and only if V is a single half-space.

Assuming h has the properties (A) and (B), we may proceed as follows. For $i = 1, 2, \cdots, M$, let k_i be the smallest integer such that $\mathcal{R}_i$ is the intersection of distinct half-spaces W_{i1}, W_{i2}, $\cdots W_{ik_i}$; i.e., k_i is the number of boundaries of the net on S_1 determined by R_i. Let K be the total number of boundaries on the net on S_1; i.e.,

$$K = \frac{1}{2} \sum_{i=1}^{M} k_i \tag{31}$$

Let B_{ij} be the portion of the boundary of W_{ij} which is a boundary of R_i. Then R_i^C can be partitioned into regions T_{i1}, T_{i2}, $\cdots$, T_{ik_i}, where each T_{ij} is bounded by B_{ij} and "hyperplanes" through $\bar{s}_i$. Hence, if we let

$$E_{i,\alpha} = R_i \cap C_{i,\alpha}^C \tag{32}$$

and

$$T_{ij,\alpha} = T_{ij} \cap C_{i,\alpha} , \tag{33}$$

we have the identity

$$\sum_{i=1}^{M} \int_{R_i} f_i d\sigma = \sum_{i=1}^{M} \left[\int_{C_{i,\alpha}} f_i d\sigma + \int_{E_{i,\alpha}} f_i d\sigma - \sum_{j=1}^{k_i} \int_{T_{ij,\alpha}} f_i d\sigma \right]. \tag{34}$$

Letting $f_i \equiv 1$, we have

$$\sigma(S_1) = \sum_{i=1}^{M}\left[\sigma(C_{i,\alpha}) + \sigma(E_{i,\alpha}) - \sum_{j=1}^{k_i} \sigma(T_{ij,\alpha})\right] \tag{35}$$

$$= M\sigma(C_\alpha) + \sum_{i=1}^{M} \sigma(E_{i,\alpha}) - \sum_{i=1}^{M}\sum_{j=1}^{k_i}\sigma(T_{ij,\alpha}) \quad . \tag{36}$$

If we next let $f_i = P_r(|\langle\bar{s},\bar{s}_i\rangle|)$ and from Eq.(30),

$\int_{T_{ij,\alpha}} P_r(|\langle\bar{s},\bar{s}_i\rangle|)\, d\sigma(\bar{s}) \geq h(\sigma(T_{ij,\alpha}))$ with equality if and only if

$T_{ij,\alpha} = C_{i,\alpha} \cap W_{ij}$.

We may then write the inequality

$$\sum_{i=1}^{M}\int_{R_i} f_i d\sigma \leq \sum_{i=1}^{M}\int_{C_{i,\alpha}} f_i d\sigma + \sum_{i=1}^{M}\int_{E_{i,\alpha}} f_i d\sigma - \sum_{i=1}^{M}\sum_{j=1}^{k_i} h(\sigma(T_{ij,\alpha})) \tag{37}$$

and, from property A, h is a strictly increasing strictly convex function. Therefore

$$\sum_{i=1}^{M}\sum_{j=1}^{k_i} h(\sigma(T_{ij,\alpha})) \geq 2K\, h\left(\frac{1}{2K}\sum_{i=1}^{M}\sum_{j=1}^{k_i}\sigma(T_{ij,\alpha})\right) \tag{38}$$

with equality if and only if $\sigma(T_{ij,\alpha})$ has the same value for all i and j.

Substituting the inequality given in (38) into Eq.(37) yields

$$\sum_{i=1}^{M}\int_{R_i} f_i d\sigma \leq \sum_{i=1}^{M}\int_{C_{i,\alpha}} f_i d\sigma + \sum_{i=1}^{M}\int_{E_{i,\alpha}} f_i d\sigma - 2K\, h\left(\frac{1}{2K}\sum_{i=1}^{M}\sum_{j=1}^{k_i}\sigma(T_{ij,\alpha})\right) \tag{39}$$

Now, from Eq.(36) we note that

$$h\left(\frac{1}{2K}\sum_{i=1}^{M}\sum_{j=1}^{k_i}\sigma(T_{ij,\alpha})\right) = h\left(\frac{1}{2K}\left[(M\sigma(C_\alpha) + \sum_{i=1}^{M}\sigma(E_{i,\alpha}) - \sigma(S_1)\right]\right). \tag{40}$$

We will now place a restriction on the $\sigma(E_{i,\alpha})$ portion of Eq.(39). Let W_1 and W_2 be two half-spaces such that

$$W_1 \cap C_{1,\alpha} \subset W_2 \cap C_{1,\alpha} \tag{41}$$

and

$$\sigma(W_1 \cap C_{1,\alpha}) = \frac{1}{2K}[M\sigma(C_\alpha) - \sigma(S_1)] \tag{42}$$

$$\sigma(W_2 \cap C_{1,\alpha}) = \frac{1}{2K}\left[M\sigma(C_\alpha) - \sigma(S_1) + \sum_{i=1}^{M}\sigma(E_{i,\alpha})\right] . \tag{43}$$

Then we may write

$$h(\sigma(W_2 \cap C_{1,\alpha})) = h(\sigma(W_1 \cap C_{1,\alpha})) + \int_A f_1 d\sigma \tag{44}$$

where

$$A = (W_2 - W_1) \cap C_{1,\alpha} \tag{45}$$

and

$$\sigma(A) = \frac{1}{2K}\sum_{i=1}^{M}\sigma(E_{i,\alpha}) . \tag{46}$$

Now, since $P_r(|\langle\bar{s},\bar{s}_i\rangle|)$ is monotone increasing in its argument, we have that

$$\int_A P_r(|\langle\bar{s},\bar{s}_i\rangle|)\, d\sigma(\bar{s}) \geq P_r(\alpha)\ \sigma(A) \Rightarrow \tag{47}$$

$$2K \int_A P_r(|\langle\bar{s},\bar{s}_i\rangle|)\, d\sigma(\bar{s}) \geq 2K\ P_r(\alpha)\ \sigma(A) \tag{48}$$

$$= \sum_{i=1}^{M} P_r(\alpha)\ \sigma(E_{i,\alpha}) \quad . \tag{49}$$

But, since $E_{i,\alpha} = R_i \cap C_{i,\alpha}^{\ C}$, we have that

$$\sum_{i=1}^{M} P_r(\alpha)\ \sigma(E_{i,\alpha}) \geq \sum_{i=1}^{M} \int_{E_{i,\alpha}} P_r(|\langle\bar{s},\bar{s}_i\rangle|)\, d\sigma(\bar{s}) \tag{50}$$

with equality if and only if $\sigma(E_{i,\alpha}) = 0$. (51)

Combining the results from (26), (39), (40), and (51), we obtain an inequality for $U(r)$

$$U(r) \leq \int_{C_{i,\alpha}} P_r(|\langle\bar{s},\bar{s}_i\rangle|)\, d\sigma(\bar{s}) - \frac{2K}{M}\, h\left(\frac{1}{2K}\ (M\sigma(C_\alpha)-\sigma(S_1))\right); \tag{52}$$

and, furthermore, there is equality if and only if such a cap size exists with the additional properties.

1. $T_{ij,\alpha} = C_{i,\alpha} \cap W_{ij}$, where W_{ij} is a half space for all i and j.
2. $\sigma(E_{i,\alpha}) = 0$ for all i.
3. $\sigma(T_{ij,\alpha}) = \frac{1}{2K}\ [M\sigma(C_\alpha)-\sigma(S_1)]$ for all i and j.

CHAPTER IV

THE CASE OF $N = 2, \quad M \geq 2$

In this Chapter the validity of properties A and B of Chapter III are proved for the case of $N = 2$. We consider the transformation which sends $(z_1, z_2) = (x_1 + jy_1,\ x_2 + jy_2)$ into (r, ρ, θ, Φ) where $z_1 = r\rho e^{j\theta}$, $z_2 = r\sqrt{1-\rho^2}\ e^{j\Phi}$, $0 \leq \rho \leq 1$, $-\pi < \theta \leq \pi$, $-\pi < \Phi \leq \pi$, and $r > 0$. The jacobian of this transformation is $r^3\rho$ so that $dm = r^3 dr d\sigma$, where $d\sigma = \rho d\rho d\theta d\Phi$. Thus, the unit sphere in C^2 has measure $\sigma(S_1) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1} \rho d\rho d\theta d\Phi = 2\pi^2.$

If $\overline{s}_0 = e^{j\gamma_0}(1,0)$ and $\overline{\rho} = \left(\rho e^{j\theta},\ \sqrt{1-\rho^2}\ e^{j\Phi}\right)$, then for $0 < \alpha < 1$ the cap equation is

$$C_\alpha = \{\overline{\rho} : |\langle \overline{\rho}, \overline{s}_0 \rangle| \geq \alpha\} = \{\overline{\rho} : \rho \geq \alpha\} \ . \tag{53}$$

For later convenience, we introduce the notation $v = \alpha^2 - \frac{1}{2}$ and $\beta = 1 - \frac{2}{M}$, then

$$\sigma(C_\alpha) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\alpha}^{1} \rho d\rho d\theta d\Phi = 2\pi^2(1-\alpha^2) = 2\pi^2(\tfrac{1}{2} - v) \quad ; \tag{54}$$

and the requirement (Eq.(28)) $\frac{1}{M}\,\sigma(S_1) \leq \sigma(C_\alpha) \leq \frac{1}{2}\,\sigma(S_1)$ becomes

$$\frac{1}{2} \leq \alpha^2 \leq 1 - \frac{1}{M} \qquad \Leftrightarrow \qquad 0 \leq v \leq \frac{1}{2}\beta \ . \tag{55}$$

Now, suppose $\overline{s}_1$ and $\overline{s}_2$ are linearly independent points on S_1; i.e., $\overline{s}_1 \neq e^{j\eta}\,\overline{s}_2$ and

$$\overline{s}_1 = e^{j\gamma_1}\left(s_1 e^{j\delta_1}, \sqrt{1 - s_1^2}\right) \tag{56}$$

$$\overline{s}_2 = e^{j\gamma_2}\left(s_2 e^{j\delta_2}, \sqrt{1 - s_2^2}\right) \quad , \tag{57}$$

then the "hyperplane" equation $|\langle\overline{\rho},\overline{s}_1\rangle| = |\langle\overline{\rho},\overline{s}_2\rangle|$ becomes

$$\rho^2 s_1^2 + (1-\rho^2)(1-s_1^2) + 2\mathrm{Re}\,\rho s_1\sqrt{1-\rho^2}\sqrt{1-s_1^2}\;e^{j(\theta-\Phi-\delta_1)} =$$

$$\rho^2 s_1^2 + (1-\rho^2)(1-s_2^2) + 2\mathrm{Re}\,\rho s_2\sqrt{1-\rho^2}\sqrt{1-s_2^2}\;e^{j(\theta-\Phi-\delta_2)} \quad . \tag{58}$$

If we let

$$\xi_1 = s_1\sqrt{1-s_1^2}\;e^{j\delta_1} \tag{59}$$

$$\xi_2 = s_2\sqrt{1-s_2^2}\;e^{j\delta_2} \tag{60}$$

and

$$\xi = \xi_1 - \xi_2 = |\xi|\,e^{j\delta} \quad , \tag{61}$$

then the "hyperplane" equation can be rewritten as

$$\sqrt{(\tfrac{1}{2})^2 - (\rho^2 - \tfrac{1}{2})^2}\;|\xi|\,\cos(\theta-\Phi-\delta) = (\rho^2 - \tfrac{1}{2})(s_2^2 - s_1^2) \quad . \tag{62}$$

If we let $t = \frac{|\xi|}{s_1^2 - s_1^2}$, and we assume without loss of generality that $s_2 \geq s_1$, then t is well-defined for $0 \leq t \leq \infty$, and we have the following cases for Eq.(62). If $t = 0$, the equation is $\rho^2 = \frac{1}{2}$; if $t = \infty$, we have either $\beta = 0$, $\beta = 1$, or $\cos(\theta-\Phi-\delta) = 0$; and if $0 < t < \infty$, we have

$$\cos(\theta-\Phi-\delta) = g_t(u) = \frac{u}{t\sqrt{(\frac{1}{2})^2 - u^2}} \qquad (63)$$

where for convenience we have let $u = \rho^2 - \frac{1}{2}$ and $g_t(u)$ is defined for $|u| \leq \tau = \frac{t}{2\sqrt{1+t^2}} \leq \frac{1}{2}$.

Proof of (A).

We now use these transformed equations to establish the convexity of h. We first let W_t be a half-space, determined by $\bar{s}_1$ and $\bar{s}_2$ and not containing $\bar{s}_0$ which intersects C_α in a set of positive measure. That is, $s_1 \leq s_2$ and

$$W_t = \{\bar{\rho}: |\langle\bar{\rho},\bar{s}_1\rangle| \geq |\langle\bar{\rho},\bar{s}_2\rangle|\} = \{\bar{\rho}: \cos(\theta-\Phi-\delta) \geq g_t(u)\} \qquad (64)$$

where for $v < \tau < \frac{1}{2}$ define

$$\omega_v(t) = \sigma(W_t \cap C_\alpha) = \int_{W_t \cap C_\alpha} d\sigma(\bar{\rho}) = \int_v^\tau k_t(u)\, du \qquad (65)$$

where

$$k_t(u) = 2\pi \arccos g_t(u) \ , \quad \tau = \frac{t}{2\sqrt{1+t^2}} \ , \quad v = \alpha^2 - \frac{1}{2} \ , \quad \text{and}$$

$$g_t(u) = \frac{u}{t\sqrt{(\frac{1}{2})^2 - u^2}}$$

then

$$\frac{\partial\omega_v(t)}{\partial t} = 2\pi \arccos g_t(\tau)\frac{d\tau}{dt} + \int_v^\tau 2\pi \frac{\partial}{\partial t}(\arccos g_t(u))\, du$$

$$= 0 + \int_v^\tau 2\pi \frac{\partial}{\partial t}(\arccos g_t(u))\, du = \frac{4\pi\tau}{t^2}\int_v^\tau \frac{u}{\sqrt{\tau^2 - u^2}} du$$

$$= \frac{4\pi\tau}{t^2}\sqrt{\tau^2 - v^2} > 0 \ . \tag{66}$$

Thus, for fixed v, $\omega_v(t)$ is a strictly increasing function of t. Now, for $0 \le \omega < \pi^2(\frac{1}{2} - v)$, which is the range of ω_v, we let $t_v(\omega)$ be the inverse function of ω_v. Next, let

$$H_v(t) = \int_v^\tau k_t(u)\, P_r(\sqrt{u + \tfrac{1}{2}})\, du \ , \tag{67}$$

and let $h_v(\omega) = H_v(t_v(\omega))$ so that $H_v(t) = h_v(\omega_v(t))$. Then

$$\frac{\partial H_v(t)}{\partial t} = \frac{\partial h_v(\omega_v(t))}{\partial\omega} \cdot \frac{\partial\omega_v(t)}{\partial t} \tag{68}$$

and

$$\frac{\partial H_v(t)}{\partial t} = \int_v^\tau \frac{\partial}{\partial t} k_t(u)\ P_r\ (\sqrt{u + \tfrac{1}{2}})\ du$$

$$= \frac{4\pi\tau}{t^2} \int_v^\tau \frac{u}{\sqrt{\tau^2 - u^2}}\ P_r(\sqrt{u + \tfrac{1}{2}})\ du \quad . \tag{69}$$

We may now write $\dfrac{\partial h_v(\omega_v(t))}{\partial\omega}$ as

$$\frac{\partial h_v(\omega_v(t)}{\partial\omega} = \frac{1}{\sqrt{\tau^2 - v^2}} \int_v^\tau \frac{u}{\sqrt{\tau^2 - u^2}}\ P_r(\sqrt{u + \tfrac{1}{2}})\ du \tag{70}$$

now integrating by parts yields.

$$\frac{\partial h_v(\omega_v(t))}{\partial\omega} = P_r(\alpha) + \int_v^\tau \sqrt{\frac{\tau^2 - u^2}{\tau^2 - v^2}}\ dP_r(\sqrt{u + \tfrac{1}{2}}) \tag{71}$$

which is a positive strictly increasing function of t, and hence of ω. Therefore, we have proved that for each fixed v, h_v is a strictly increasing strictly convex function.

<u>Proof of (B)</u>.

We now prove the conjecture that if V is the intersection of a finite number of half spaces, at least one of which does not contain $\bar{s}_i$, then

$$\int_{C_{i,\alpha} \cap V} P_r(|\langle \bar{s}, \bar{s}_i \rangle|)\ d\sigma(\bar{s}) \geq h(\sigma(C_{i,\alpha} \cap V))$$

with equality if and only if V is a single half space. We proceed by recalling that $k_t(u) = 2\pi \arccos g_t(u)$ where $g_t(u) = \dfrac{u}{t\sqrt{\frac{1}{4} - u^2}}$.

Therefore, we observe that

$$\frac{\partial k_t(u)}{\partial u} = \frac{-\pi}{2(\frac{1}{4} - u^2)\sqrt{(\frac{t}{2})^2 - (t^2 + 1)\, u^2}}$$

$$= \frac{-\pi\tau}{t(\frac{1}{4} - u^2)\sqrt{\tau^2 - u^2}} \tag{72}$$

which for fixed u is a strictly increasing function of t and is always negative. Now, let $W_{t_1}, W_{t_2}, \cdots, W_{t_n}$ be half-spaces such that $V = \bigcap_{i=1}^{n} W_{t_i}$ intersects C_α in a set of positive measure and $\bar{s}_0 \notin W_{t_i}$ for $i = 1, 2, \cdots, m \leq n$.

Now, define $g_t(u) = 1$ and $k_t(u) = 0$ for $\tau < u < \frac{1}{2}$. Then

$$\sigma(W_t \cap C_\alpha) = \begin{cases} \int_V^{\frac{1}{2}} k_t(u)\, du & \bar{s}_0 \notin W_t \\ \int_V^{\frac{1}{2}} [2\pi^2 - k_t(u)] du & \bar{s}_0 \in W_t \end{cases} \tag{73}$$

Therefore,

$$\sigma(V \cap C_\alpha) = \int_V^{\frac{1}{2}} k(u)\, du \tag{74}$$

where we describe $k(u)$ as follows. Let

$$d(i) = \begin{cases} +1 & i = 1, 2, \cdots, m \\ -1 & i = m+1, \cdots, n \end{cases} , \tag{75}$$

then there is a partition $u_0 < u_1 < \cdots < u_k$ of $[v,\frac{1}{2}]$ such that for $u \in [u_{j-1}, u_j]$,

$$k(u) = \lambda + \frac{1}{2} \sum d(i_\ell)\, k_{t_{i_\ell}}(u) \tag{76}$$

where λ is a constant and (i_ℓ) is a collection of not necessarily distinct elements of $\{1, 2, \cdots, n\}$ such that at most two of them belong to $\{1, 2, \cdots, m\}$.

In particular, this description shows that k is continuous on $[v,\frac{1}{2}]$, differentiable in (u_{j-1}, u_j) and has right- and left-hand derivatives at the left and right end points, respectively. In fact, these derivaties are given by

$$\frac{dk(u)}{du} = \frac{1}{2} \sum d(i_\ell) \frac{\partial k_{t_{i_\ell}}(u)}{\partial u} . \tag{77}$$

If we now let W_t be a half space such that $\bar{s}_0 \notin W_t$ and $\sigma(W_t \cap C_\alpha) = \sigma(V \cap C_\alpha)$, then $\omega_v(t) = \sigma(W_t \cap C_\alpha) = \sigma(V \cap C_\alpha) \le \sigma(W_{t_i} \cap C_\alpha) = \omega(t_i)$ for $i = 1, 2, \cdots, m \rightarrow t_i > t$ for $i = 1, 2, \cdots, m$. Consequently, from Eq.(72)

$$\frac{dk(u)}{du} \ge \frac{\partial k_t(u)}{\partial u} \tag{78}$$

for $v \le u \le \tau$ with equality if and only if $V = W_{t_i}$ for some i

with $t_i = t$ and $1 \le i \le m$. But $\sigma(W_t \cap C_\alpha) = \sigma(V \cap C_\alpha) =$

$\int_v^{\frac{1}{2}} k(u)du = \int_v^{\frac{1}{2}} k_t(u)\, du$ and $\frac{dk(u)}{du} \ge \frac{\partial k_t(u)}{\partial u}$ for $v \le u \le \tau$ implies

there is a point $u_0 \in [v, \frac{1}{2}]$ such that $k_t(u) \ge k(u)$ for $u \le u_0$ and $k_t(u) \le k(u)$ for $u \ge u_0$. Since P_r is monotone increasing, we may therefore write

$$\int_{V \cap C_\alpha} P_r(|\langle\overline{\rho}, \overline{s}_0\rangle|)\, d\sigma(\overline{\rho}) - h_v(\omega_v(t)) = \int_v^{\frac{1}{2}} [k(u)-k_t(u)]\; P_r(\sqrt{u + \tfrac{1}{2}})\, du$$

$$= \int_v^{u_0} [k(u)-k_t(u)]\; P_r(\sqrt{u + \tfrac{1}{2}})\, du + \int_{u_0}^{\frac{1}{2}} [k(u)-k_t(u)]\; P_r(\sqrt{u + \tfrac{1}{2}})\, du$$

$$\ge P_r\sqrt{u_0 + \tfrac{1}{2}}\left[\int_v^{u_0} [k(u)-k_t(u)]\, du + \int_{u_0}^{\frac{1}{2}} [k(u)-k_t(u)]\, du\right] = 0 \qquad (79)$$

with equality if and only if $k(u) = k_t(u)$ for all $u \in [v, \frac{1}{2}]$. Hence

$$\int_{V \cap C_\alpha} P_r(|\langle\overline{\rho}, \overline{s}_0\rangle|)\, d\sigma(\overline{\rho}) \ge h_v(\sigma(V \cap C_\alpha)) \qquad (80)$$

with equality if and only if V is a single half-space.

CHAPTER V

SOME NECESSARY CONDITIONS

In this Chapter we establish some necessary conditions that must be satisfied for the existence of an optimal signal set. We consider the case in which $k_i \geq 2$ for $i = 1, 2, \cdots, M \rightarrow 2K \geq 2M \Leftrightarrow x = \frac{M\pi}{2K} \leq \frac{\pi}{2}$.

For a given allowable cap size, i.e., $0 \leq v \leq \frac{1}{2}\beta$, one such necessary condition is the existence of a half-space W_t such that

$$\sigma(W_t \cap C_\alpha) = \frac{1}{2K}\left[M\sigma(C_\alpha) - \sigma(S_1)\right] \quad . \tag{81}$$

We now define

$$W_x(v) = \frac{1}{2K}\left[M\sigma(C_\alpha) - \sigma(S_1)\right] = 2\pi x\left(\frac{1}{2}\beta - v\right) \tag{82}$$

which is in the domain of t_v, the inverse function of ω_v. Hence, we define

$$T_x(v) = t_v(W_x(v)) \tag{83}$$

then

$$\omega_v(T_x(v)) = W_x(v) \quad . \tag{84}$$

Thus, the half-space determined by $t = T_x(v)$ must satisfy the necessary condition

$$\frac{\partial W_X(v)}{\partial v} = \frac{\partial \omega_v(T_X(v))}{\partial t} \cdot \frac{\partial T_X(v)}{\partial v} + \frac{\partial \omega_v(T_X(v))}{\partial v} \qquad (85)$$

Now, from Eq.(82), we have that

$$\frac{\partial W_x(v)}{\partial v} = -\ 2\pi x \quad . \qquad (86)$$

Now, for convenience of notation, we define

$$\tau_x(v) = \frac{T_x(v)}{2\sqrt{1 + T_x^{\ 2}(v)}} \quad . \qquad (87)$$

Then we may write

$$\omega_v(T_x(v)) = \int_v^{\tau_x(v)} k_{T_x(v)}(u)\ du \qquad (88)$$

which yields

$$\frac{\partial \omega_v(T_x(v))}{\partial v} = -\ k_{T_x(v)}(v) = -\ 2\pi \arccos g_{T_x(v)}(v) \quad ; \qquad (89)$$

and, from Eq.(66) we have that

$$\frac{\partial \omega_v(T_x(v))}{\partial t} = \frac{4\pi\tau_x(v)}{T_x^{\ 2}(v)} \sqrt{\tau_x^{\ 2}(v) - v^2} \quad . \qquad (90)$$

Now, substituting Eqs.(86), (89), and (90) into Eq.(85) and solving for $\frac{\partial T_x(v)}{\partial v}$, we obtain

$$\frac{\partial T_x(v)}{\partial v} = \frac{T_x^{\,2}(v)[\operatorname{arcos} g_{T_x(v)}(v) - x]}{2\tau_x(v)\sqrt{\tau_x^{\,2}(v) - v^2}} \quad . \tag{91}$$

Now, from Eq.(52), we have

$$\begin{aligned} U_x(v) &= \int_{C_\alpha} P_r(|\langle\overline{\rho},\overline{s}_0\rangle|)\, d\sigma(\overline{\rho}) - \frac{2K}{M}\, h_v\!\left(\frac{1}{2K}\,[M\sigma(C_\alpha)-\sigma(S_1)]\right) \\ &= 2\pi^2 \int_v^{\frac{1}{2}} P_r\!\left(\sqrt{u+\tfrac{1}{2}}\right) du - \frac{\pi}{x}\, h_v(W_x(v)) \\ &= 2\pi^2 \int_v^{\frac{1}{2}} P_r\!\left(\sqrt{u+\tfrac{1}{2}}\right) du - \frac{\pi}{x}\, H_V(T_x(v)) \quad . \end{aligned} \tag{92}$$

Then,

$$\begin{aligned} \frac{\partial U_x(v)}{\partial v} &= -\,2\pi^2 P_r(\alpha) - \frac{\pi}{x}\left[\frac{\partial H_V(T_x(v))}{\partial v} + \frac{\partial H_V(T_x(v))}{\partial t}\,\frac{\partial T_x(v)}{\partial v}\right] \\ &= \frac{2\pi^2}{x}\left[x - \operatorname{arcos} g_{T_x(v)}(v)\right] \int_v^{\tau_x(v)} \sqrt{\frac{\tau_x^{\,2}(v) - u^2}{\tau_x^{\,2}(v) - v^2}}\; dP_r\!\left(\sqrt{u+\tfrac{1}{2}}\right). \end{aligned} \tag{93}$$

Another property of the $U_x(v)$ equation is obtained with the aid of the convexity property of h_v. This is that

$$\frac{\partial U_x(v)}{\partial x} = \frac{\pi}{x^2}\left[h_v(W_x(v)) - \frac{\partial h_v(W_x(v))}{\partial \omega}\cdot W_x(v)\right] \le 0 \quad . \tag{94}$$

If we now consider requirement 2, Chapter III, i.e.,

$\sigma(E_{i,\alpha}) = \sigma(R_i \cap C_{i,\alpha}{}^C) = 0$, we find some additional requirement for the existence of the cap is the existence of a $v = \alpha^2 - \frac{1}{2}$, $0 \le v \le \frac{1}{2}\beta$ such that

$$2 \arccos g_{T_X(v)}(v) = \frac{2\pi}{2K/M} \qquad \Rightarrow$$

$$\arccos g_{T_X(v)}(v) = x \qquad . \tag{95}$$

We shall show, in fact, that there is exactly one such v, call it $V(x)$, and $V(x)$ is the unique point at which the maximum of T_x and the minimum of U_x occurs in the interval $[0, \frac{1}{2}\beta]$.

First of all, for $x = \frac{\pi}{2}$, $g_{T_x(v)}(v) = \cos x \Leftrightarrow v = 0$, so we let $V(\frac{\pi}{2}) = 0$; for $x < \frac{\pi}{2}$ we have

$$g_{T_x(v)}(v) = \cos x \qquad \Leftrightarrow$$

$$T_x(v) = \frac{v}{\cos x \sqrt{(\frac{1}{2})^2 - v^2}} \qquad . \tag{96}$$

On the other hand,

$$\omega_v(t) = \int_v^\tau k_t(u)\, du$$

$$= 2\pi \int_{\arccos(2v)}^{\arccos(2\tau)} \arccos\left[\frac{1}{t \tan \Omega}\right] d(\tfrac{1}{2}\cos \Omega) \qquad ; \tag{97}$$

and integrating by parts, we obtain

$$\omega_v(t) = 2\pi[\frac{1}{2} \text{ arcos } j_t(v) - v \text{ arcos } g_t(v)] \tag{98}$$

where

$$j_t(v) = \frac{1}{2\sqrt{1 + t^2}\sqrt{(\frac{1}{2})^2 - v^2}} \quad . \tag{99}$$

Hence, the defining equation for $T_x(v)$

$$\omega_v(T_x(v)) = W_x(v)$$

becomes

$$\frac{1}{2} \text{ arcos } j_{T_x(v)}(v) - v \text{ arcos } g_{T_x(v)}(v) = x(\frac{1}{2}\beta - v) \quad . \tag{100}$$

Thus, we are looking for a value of v which satisfies the system

$$\frac{v}{T_x(v)\sqrt{(\frac{1}{2})^2 - v^2}} = \cos x \quad . \tag{101}$$

$$\frac{1}{2\sqrt{1 + T_x^2(v)}\sqrt{(\frac{1}{2})^2 - v^2}} = \cos \beta x \quad . \tag{102}$$

The solution is easily found to be

$$V(x) = \frac{1}{2}\frac{\tan \beta x}{\tan x} \tag{103}$$

and

$$T_x(V(x)) = \frac{\sin \beta x}{\sqrt{\cos^2 \beta x - \cos^2 x}} \quad . \tag{104}$$

To show that $V(x) \le \frac{1}{2}\beta$ for $0 < x < \frac{\pi}{2}$, we consider $2 \tan x\ (\frac{1}{2}\beta - V(x)) = \beta \tan x - \tan \beta x$. This function is 0 for $x = 0$ and has derivative $\beta(\sec^2 x - \sec^2 \beta x)$ and since $0 < \beta \le 1$

$$\beta(\sec^2 x - \sec^2 \beta x) \ge 0 \Rightarrow$$

$$\beta \tan x - \tan \beta x \ge 0 \quad . \tag{105}$$

Finally, if $U_0 = U_x(V(x))$ is the value of U_x which will be attained if an optimal signal set occurs for a given value of x, we have $U_0'(x) = \frac{\partial U_x(V(x))}{\partial x} \le 0$. Thus, for fixed M, U_0 is a decreasing function of x; and, hence, the maximum possible value of U_0 is obtained for $\frac{2K}{M} = M - 1$, i.e., $x = \frac{\pi}{M-1}$.

CHAPTER VI

A TRANSFORMATION INTO THREE-SPACE

In this Chapter we perform a transformation that maps the unit sphere in C^2 onto the unit sphere in three-dimensional real Euclidean space. In particular, we apply this transformation to the hyperplane equation (Eq.(62)) and to the equation for the boundary of the cap.

The equation (Eq.(62)) of the hyperplane between two signals, say $\bar{s}_1$ and $\bar{s}_2$, is

$$|\langle\bar{\rho},\bar{s}_1\rangle| = |\langle\bar{\rho},\bar{s}_2\rangle| \Leftrightarrow \sqrt{(\tfrac{1}{2})^2-(\rho^2-\tfrac{1}{2})^2}\ |\xi|\ \cos(\theta-\Phi-\delta) = (\rho^2-\tfrac{1}{2})(s_2{}^2-s_1{}^2) \Leftrightarrow$$

$$\sqrt{\tfrac{1}{4}-u^2}\ |\xi|\ \cos(\psi) = u(s_2{}^2-s_1{}^2) \qquad (106)$$

where we have defined $u = \rho^2 - \frac{1}{2}$ and $\psi = \theta-\Phi-\delta$. The equation for the boundary of the cap about signal $\bar{s}_1$ is

$$|\langle\bar{\rho},\bar{s}_1\rangle| = \alpha \Leftrightarrow 2(\tfrac{1}{2}-s_1{}^2)(\tfrac{1}{2}-\rho^2) + 2\rho s_1\sqrt{1-s_1{}^2}\sqrt{1-\rho^2}\ \cos\psi = \alpha^2-\tfrac{1}{2}$$

$$\Leftrightarrow 2s'u + 2\sqrt{\tfrac{1}{4}-s'^2}\sqrt{\tfrac{1}{4}-u^2}\ \cos\psi = v \qquad (107)$$

where $v = \alpha^2 - \frac{1}{2}$ and $s' = s_1{}^2 - \frac{1}{2}$.

Now, Eqs.(106) and (107) are for a given cap size (α) and a given set of signal vectors, functions only of the two variables $\chi = \theta-\Phi$ and u. Solving for u as a function of χ, we obtain from (106)

$$u = \frac{|\xi| \cos(\chi-\delta)}{2\sqrt{s_2^{\,2}-s_1^{\,2} + |\xi|^2 \cos^2(\chi-\delta)}} \tag{108}$$

and from (107)

$$u = \frac{vs' \pm \sqrt{v^2 s'^2 - (s'^2 + (\frac{1}{4} - s'^2) \cos^2(\chi-\delta))(v^2 - \frac{1}{2})}}{2s'^2 + (\frac{1}{4} - s'^2) \cos^2(\chi-\delta)} \tag{109}$$

Using the coordinate transformation shown in Figure VI-1 for $\delta = 0$, we transform the u, χ equations into points on the unit sphere in three real dimensions with

$$\left.\begin{aligned} x &= 2u \\ y &= \sqrt{1 - 4u^2} \quad \cos(\chi) \\ z &= \sqrt{1 - 4u^2} \quad \sin(\chi) \end{aligned}\right\} \tag{110}$$

In Section VII, Eqs.(108) and (109) are used to illustrate the configurations of the caps and hyperplanes. These figures are then transformed by (110) and plotted on the surface of a unit sphere in three dimensions.

Substituting the transformation (110) into the hyperplane Eq.(106), we see that the hyperplane for $\delta = 0$ is given by $x = ty$ where $t = \frac{|\xi|}{s_2^{\,2}-s_1^{\,2}}$ and the intersection with the unit sphere is given by the set of equations

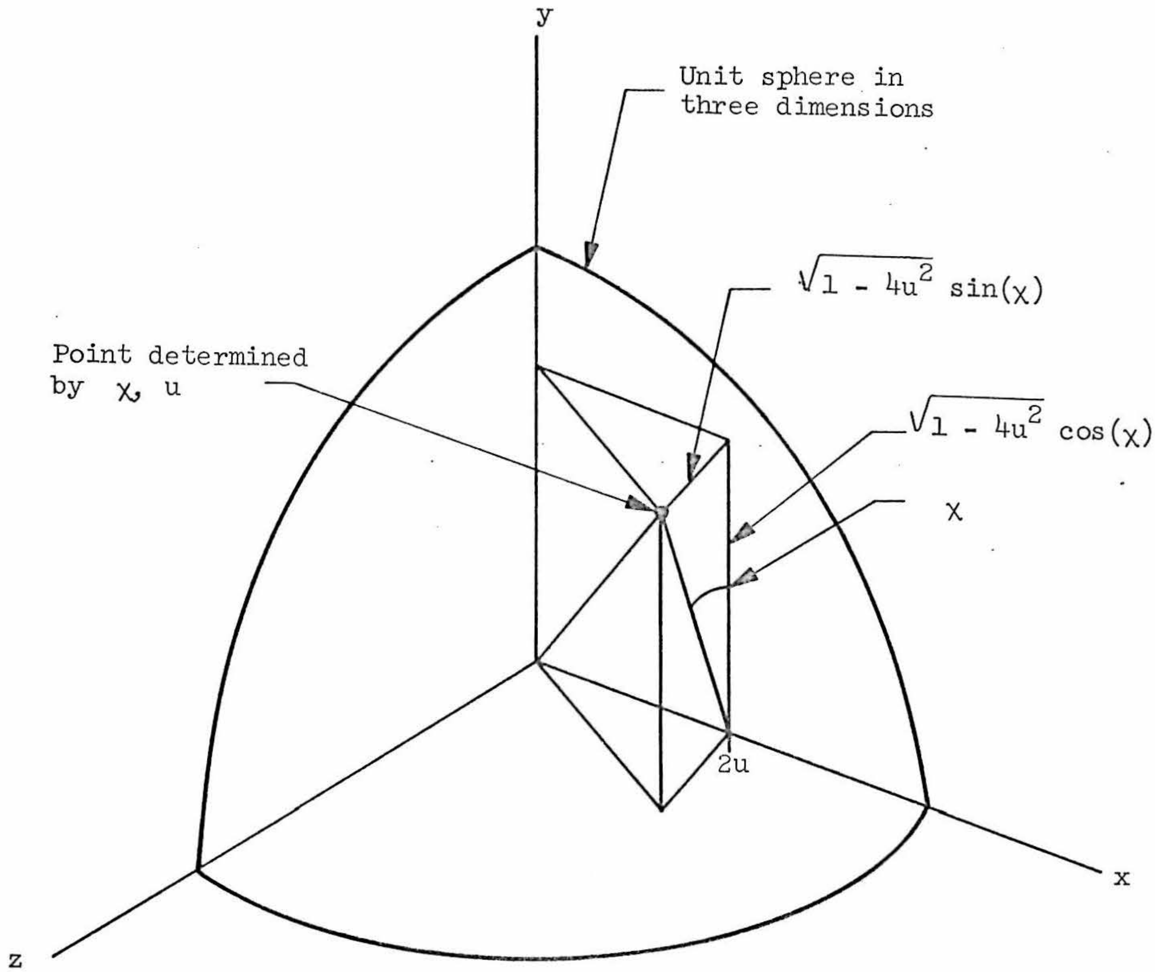

FIGURE VI-1

The co-ordinate transformation from the unit sphere in C^2 onto the unit sphere in three-dimensional real space ($\delta = 0$).

$$x = ty$$
$$z = \sqrt{1 - x^2}\ \sin\psi \qquad (111)$$

That is, the hyperplane in C^2 is transformed into a two-dimensional real plane which intersects the sphere and passes through the origin. The decision regions in three dimensional real space are now the intersection of half spaces determined by these transformed hyperplanes. Let H_{ij}^t denote the hyperplane in three-space between $\bar{s}_i$ and $\bar{s}_j$. Then let W_{ij}^t be the half-space containing $\bar{s}_i$. The decision region $\mathcal{R}_i^t$ is then the intersection of M-1 of these half spaces, i.e.

$$\mathcal{R}_i^t = \bigcap_{k \neq i} W_{ik}^t \qquad (112)$$

and is therefore a convex region bounded by a certain number of hyperplanes that pass through the origin. Now, by using this transformation, we note that the maximum likelihood regions forming the net on the three-dimensional sphere can be composed of regular spherical polygons. Since the $\mathcal{R}_i^t$ are convex, a vertex on the surface of the sphere must be formed by at least three edges. Thus, we have that $3V \leq 2K$ where V is the total number of vertices on the net and K is the total number of edges on the net. We may now apply Euler's formula [5], $V-K+M = 2$, for the net and obtain an inequality for K

$$K \leq 3(M-2) \qquad (113)$$

Since we require the same number of boundaries on each of the decision regions, the total number of boundaries K must be an integer (I) times the total number of signals divided by 2; i.e.,

$$K = \frac{IM}{2} \quad . \tag{114}$$

We may therefore rewrite (65) as an inequality for I or M as

$$I \le \frac{6M - 12}{M} \tag{115}$$

or

$$M \ge \frac{12}{6 - I} \quad . \tag{116}$$

Now, as shown in Chapter V, $U_0(r)$ Eq.(52) is a monotone decreasing function of x and hence a monotone increasing function in K. Therefore, for a given M, we wish to choose K and thus I as large as possible. Consequently, we wish to obtain equality in inequalities (115) and (116) since these codes will have the maximum number of boundaries for each decision region and the same number of boundaries for each region. The only cases of equality are $M = 3, 4, 6$, and 12. The corresponding values of K are $K = 3, 6, 12, 30$. When $M = 2$, the above inequalities do not hold since there are no vertices.

CHAPTER VII

SOLUTIONS FOR $N = 2$

In this Chapter we use the results of Chapters IV, V, and VI to construct the globally optimum signal sets for $N = 2$, $M = 2, 3, 4,$ 6, and 12. We obtain the value of $|\langle \bar{s}_i, \bar{s}_j \rangle|$ for all i, j for each of these cases. We then graphically present the results for these cases showing the location of the signal vectors, the caps, and the hyperplanes.

Two Signals:

If $M = 2$, then $\beta = 0$, and the requirement $0 \le v \le \frac{1}{2}\beta \Rightarrow v = 0$ and $\alpha = \frac{1}{\sqrt{2}}$. Also, $1 \le \frac{2K}{M} \le M-1 \Leftrightarrow K = 1$. Thus the decision region for $\bar{s}_1$ must be

$$\{\bar{\rho}: |\langle \bar{\rho}, \bar{s}_1 \rangle| \ge |\langle \bar{\rho}, \bar{s}_0 \rangle|\} = \{\bar{\rho}: \rho^2 \le \tfrac{1}{2}\} \quad \Leftrightarrow \tag{117}$$

$t = \dfrac{s_1}{\sqrt{1-s_1^{\,2}}} = 0 \Leftrightarrow s_1 = 0$. That is, $\bar{s}_0 = e^{i\gamma_0}(1,0)$ and $\bar{s}_1 = e^{i\gamma_1}(0,1)$

and hence $\langle \bar{s}_0, \bar{s}_1 \rangle = 0$.

Three Signals:

$$M = 3, \quad K = 3 \Rightarrow \quad x = \frac{\pi}{2}, \quad \beta = \frac{1}{3}$$

Then $V(x) = 0$ and $\alpha = \frac{1}{\sqrt{2}}$, $t = T_x(V(x)) = \dfrac{\sin \beta x}{\sqrt{\cos^2 \beta x - \cos^2 x}} = \tan\frac{\pi}{6} =$

$\frac{1}{\sqrt{3}}$. Hence $\frac{s_1}{\sqrt{1-s_1^2}} = \frac{s_2}{\sqrt{1-s_2^2}} = t \Leftrightarrow s_1 = s_2 = \frac{\sin \beta x}{\sin x} = \frac{1}{2}$, and there-

$\bar{s}_0 = e^{i\gamma_0}(1,0)$ $\bar{s}_1 = e^{i\gamma_1}(\frac{1}{2} e^{i(\delta + \frac{\pi}{2})}, \frac{1}{2}\sqrt{3})$, and $\bar{s}_2 = e^{i\gamma_2}(\frac{1}{2} e^{i(\delta+\frac{3\pi}{2})}$,

$\frac{1}{2}\sqrt{3})$, resulting in $|\langle \bar{s}_0, \bar{s}_1\rangle| = |\langle \bar{s}_0, \bar{s}_2\rangle| = |\langle \bar{s}_1, \bar{s}_2\rangle| = \frac{1}{2}$.

Four Signals:

$$M = 4, \quad K = 6 \Rightarrow x = \frac{\pi}{3} \ , \ \beta = \frac{1}{2}$$

Then, $V(x) = \frac{1}{2} \frac{\tan \pi/6}{\tan \pi/3} = \frac{1}{6}$ and $\alpha = \sqrt{\frac{2}{3}}$, $t = \frac{\sin \pi/6}{\sqrt{\cos^2 \frac{\pi}{6} - \cos^2 \frac{\pi}{3}}} = \frac{1}{\sqrt{2}}$

$$s_0 = 1$$

$$s_1 = s_2 = s_3 = \frac{\sin \pi/6}{\sin \pi/3} = \frac{1}{\sqrt{3}}$$

Therefore we have

$$\bar{s}_0 = e^{i\gamma_0}(1,0) \ , \ \bar{s}_1 = e^{i\gamma_1}\left(\frac{1}{\sqrt{3}} e^{i(\delta)} , \sqrt{\frac{2}{3}}\right)$$

$$\bar{s}_2 = e^{i\gamma_2}\left(\frac{1}{\sqrt{3}} e^{i(\delta+\frac{2\pi}{3})}, \sqrt{\frac{2}{3}}\right), \text{ and } \bar{s}_3 = e^{i\gamma_3}\left(\frac{1}{\sqrt{3}} e^{i(\delta + \frac{4\pi}{3})}, \sqrt{\frac{2}{3}}\right).$$

resulting in

$$|\langle\bar{s}_0,\bar{s}_1\rangle| = |\langle\bar{s}_0,\bar{s}_2\rangle| = |\langle\bar{s}_0,\bar{s}_3\rangle| = |\langle\bar{s}_1,\bar{s}_2\rangle| = |\langle\bar{s}_1,\bar{s}_3\rangle| =$$

$$|\langle\bar{s}_2,\bar{s}_3\rangle| = \frac{1}{\sqrt{3}}$$

Six Signals:

$$M = 6, \quad K = 12 \Rightarrow x = \frac{\pi}{4} \quad , \quad \beta = \frac{2}{3}$$

$$V(x) = \frac{1}{2}\frac{\tan \pi/6}{\tan \pi/4} = \frac{1}{2\sqrt{3}} \quad \text{and} \quad \alpha = \sqrt{\frac{1+\sqrt{3}}{2\sqrt{3}}} \qquad t = \frac{\sin \pi/6}{\sqrt{\cos^2 \pi/6 - \cos^2 \pi/4}} = 1$$

$$s_0 = 1$$

$$s_1 = s_2 = s_3 = s_4 = \frac{\sin \pi/6}{\sin \pi/4} = \frac{1}{\sqrt{2}}$$

Select $s_5 = 0$.

Therefore we have the following set of signals.

$$\bar{s}_0 = e^{i\gamma_0}(1,0) \quad , \quad \bar{s}_1 = e^{i\gamma_1}\left(\frac{1}{\sqrt{2}}\, e^{i(\delta + \frac{\pi}{4})} \, , \, \frac{1}{\sqrt{2}}\right)$$

$$\bar{s}_2 = e^{i\gamma_2}\left(\frac{1}{\sqrt{2}}\, e^{i(\delta + \frac{3\pi}{4})} \, , \, \frac{1}{\sqrt{2}}\right), \quad \bar{s}_3 = e^{i\gamma_3}\left(\frac{1}{\sqrt{2}}\, e^{i(\delta + \frac{5\pi}{4})} \, , \, \frac{1}{\sqrt{2}}\right)$$

$$\bar{s}_4 = e^{i\gamma_4}\left(\frac{1}{\sqrt{2}}\, e^{i(\delta + \frac{7\pi}{4})} \, , \, \frac{1}{\sqrt{2}}\right) \quad , \quad \bar{s}_5 = e^{i\gamma_5}(0,1)$$

resulting in

$$|\langle\bar{s}_0,\bar{s}_1\rangle| = |\langle\bar{s}_0,\bar{s}_2\rangle| = |\langle\bar{s}_0,\bar{s}_3\rangle| = |\langle\bar{s}_0,\bar{s}_4\rangle| = |\langle\bar{s}_1,\bar{s}_2\rangle| =$$
$$|\langle\bar{s}_1,\bar{s}_3\rangle| = |\langle\bar{s}_1,\bar{s}_4\rangle| = |\langle\bar{s}_1,\bar{s}_5\rangle| = |\langle\bar{s}_2,\bar{s}_3\rangle|$$
$$= |\langle\bar{s}_2,\bar{s}_4\rangle| = |\langle\bar{s}_2,\bar{s}_5\rangle| = |\langle\bar{s}_3,\bar{s}_4\rangle| = |\langle\bar{s}_3,\bar{s}_5\rangle| = |\langle\bar{s}_4,\bar{s}_5\rangle| = \frac{1}{\sqrt{2}}$$

and $|\langle\bar{s}_0,\bar{s}_5\rangle| = 0.$

Twelve Signals:

$$M = 12, \quad K = 30 \Rightarrow X = \frac{\pi}{5}, \quad \beta = \frac{5}{6}$$

$$V(x) = \frac{1}{2}\ \frac{\tan \pi/6}{\tan \pi/5} = \frac{1}{2}\sqrt{\frac{3+\sqrt{5}}{3(5-\sqrt{5})}} \approx 0.39733$$

and $$\alpha = \sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{3+\sqrt{5}}{3(5-\sqrt{5})}}\right)} \approx 0.94727$$

$$t = \frac{\sin \pi/6}{\sqrt{\cos^2 \pi/6 - \cos^2 \pi/5}} = \sqrt{\frac{2}{3-\sqrt{5}}} \approx 1.6180$$

then $s_0 = 1$

$$s_1 = s_2 = s_3 = s_4 = s_5 = \frac{\sin \pi/6}{\sin \pi/5} = \sqrt{\frac{2}{5-\sqrt{5}}} \approx 0.85065$$

$$s_6 = s_7 = s_8 = s_9 = s_{10} = \frac{\sqrt{\sin^2 x - \sin^2 \beta x}}{\sin x} = \frac{\sqrt{\sin^2 \pi/5 - \sin^2 \pi/6}}{\sin \pi/5}$$

$$= \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \approx 0.52573$$ and $s_{11} = 0.$

The optimum signal set is therefore

$$\bar{s}_0 = e^{i\gamma}(1,0) \quad ,$$

$$\bar{s}_1 = e^{i\gamma_1}\left(\sqrt{\frac{2}{5-\sqrt{5}}}\, e^{i(\delta + \frac{\pi}{5})} , \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\right) \quad ,$$

$$\bar{s}_2 = e^{i\gamma_2}\left(\sqrt{\frac{2}{5-\sqrt{5}}}\, e^{i(\delta + \frac{3\pi}{5})} , \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\right) \quad ,$$

$$\bar{s}_3 = e^{i\gamma_3}\left(\sqrt{\frac{2}{5-\sqrt{5}}}\, e^{i(\delta + \pi)} , \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\right) \quad ,$$

$$\bar{s}_4 = e^{i\gamma_4}\left(\sqrt{\frac{2}{5-\sqrt{5}}}\, e^{i(\delta + \frac{7\pi}{5})} , \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\right) \quad ,$$

$$\bar{s}_5 = e^{i\gamma_5}\left(\sqrt{\frac{2}{5-\sqrt{5}}}\, e^{i(\delta + \frac{9\pi}{5})} , \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\right) \quad ,$$

$$\bar{s}_6 = e^{i\gamma_6}\left(\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{3}}}\, e^{i(\delta)} , \sqrt{\frac{2}{5-\sqrt{5}}}\right) \quad ,$$

$$\bar{s}_7 = e^{i\gamma_7}\left(\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\, e^{i(\delta + \frac{2\pi}{5})} , \sqrt{\frac{2}{5-\sqrt{5}}}\right) \quad ,$$

$$\bar{s}_8 = e^{i\gamma_8}\left(\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\, e^{i(\delta + \frac{4\pi}{5})} , \sqrt{\frac{2}{5-\sqrt{5}}}\right) ,$$

$$\bar{s}_9 = e^{i\gamma_9}\left(\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\, e^{i(\delta + \frac{6\pi}{5})} , \sqrt{\frac{2}{5-\sqrt{5}}}\right) ,$$

$$\bar{s}_{10} = e^{i\gamma_{10}} \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\, e^{i(\gamma + \frac{8\pi}{5})} , \sqrt{\frac{2}{5-\sqrt{5}}} ,$$

and

$$\bar{s}_{11} = e^{i\gamma_{11}}(0,1)$$

which results in

$$|\langle\bar{s}_0,\bar{s}_1\rangle| = |\langle\bar{s}_0,\bar{s}_2\rangle| = |\langle\bar{s}_0,\bar{s}_3\rangle| = |\langle\bar{s}_0,\bar{s}_4\rangle| = |\langle\bar{s}_0,\bar{s}_5\rangle| =$$

$$|\langle\bar{s}_1,\bar{s}_2\rangle| = |\langle\bar{s}_1,\bar{s}_5\rangle| = |\langle\bar{s}_1,\bar{s}_6\rangle| = |\langle\bar{s}_1,\bar{s}_7\rangle| = |\langle\bar{s}_2,\bar{s}_3\rangle| =$$

$$|\langle\bar{s}_2,\bar{s}_7\rangle| = |\langle\bar{s}_2,\bar{s}_8\rangle| = |\langle\bar{s}_3,\bar{s}_4\rangle| = |\langle\bar{s}_3,\bar{s}_8\rangle| = |\langle\bar{s}_3,\bar{s}_9\rangle| =$$

$$|\langle\bar{s}_4,\bar{s}_5\rangle| = |\langle\bar{s}_4,\bar{s}_9\rangle| = |\langle\bar{s}_4,\bar{s}_{10}\rangle| = |\langle\bar{s}_5,\bar{s}_6\rangle| = |\langle\bar{s}_5,\bar{s}_{10}\rangle| =$$

$$|\langle\bar{s}_6,\bar{s}_7\rangle| = |\langle\bar{s}_6,\bar{s}_{10}\rangle| = |\langle\bar{s}_6,\bar{s}_{11}\rangle| = |\langle\bar{s}_7,\bar{s}_8\rangle| = |\langle\bar{s}_7,\bar{s}_{11}\rangle| =$$

$$|\langle\bar{s}_8,\bar{s}_9\rangle| = |\langle\bar{s}_8,\bar{s}_{11}\rangle| = |\langle\bar{s}_9,\bar{s}_{10}\rangle| = |\langle\bar{s}_9,\bar{s}_{11}\rangle| = |\langle\bar{s}_{10},\bar{s}_{11}\rangle| =$$

$$\sqrt{\frac{2}{5-\sqrt{5}}} \quad ;$$

$$|\langle \bar{s}_0, \bar{s}_6 \rangle| = |\langle \bar{s}_0, \bar{s}_7 \rangle| = |\langle \bar{s}_0, \bar{s}_8 \rangle| = |\langle \bar{s}_0, \bar{s}_9 \rangle| = |\langle \bar{s}_0, \bar{s}_{10} \rangle| =$$

$$|\langle \bar{s}_1, \bar{s}_3 \rangle| = |\langle \bar{s}_1, \bar{s}_4 \rangle| = |\langle \bar{s}_1, \bar{s}_8 \rangle| = |\langle \bar{s}_1, \bar{s}_{10} \rangle| = |\langle \bar{s}_1, \bar{s}_{11} \rangle| =$$

$$|\langle \bar{s}_2, \bar{s}_4 \rangle| = |\langle \bar{s}_2, \bar{s}_5 \rangle| = |\langle \bar{s}_2, \bar{s}_6 \rangle| = |\langle \bar{s}_2, \bar{s}_9 \rangle| = |\langle \bar{s}_2, \bar{s}_{11} \rangle| =$$

$$|\langle \bar{s}_3, \bar{s}_5 \rangle| = |\langle \bar{s}_3, \bar{s}_7 \rangle| = |\langle \bar{s}_3, \bar{s}_{10} \rangle| = |\langle \bar{s}_3, \bar{s}_{11} \rangle| = |\langle \bar{s}_4, \bar{s}_6 \rangle| =$$

$$|\langle \bar{s}_4, \bar{s}_8 \rangle| = |\langle \bar{s}_4, \bar{s}_{11} \rangle| = |\langle \bar{s}_5, \bar{s}_7 \rangle| = |\langle \bar{s}_5, \bar{s}_9 \rangle| = |\langle \bar{s}_5, \bar{s}_{11} \rangle| =$$

$$|\langle \bar{s}_6, \bar{s}_8 \rangle| = |\langle \bar{s}_6, \bar{s}_9 \rangle| = |\langle \bar{s}_7, \bar{s}_9 \rangle| = |\langle \bar{s}_7, \bar{s}_{10} \rangle| = |\langle \bar{s}_8, \bar{s}_{10} \rangle| =$$

$$= \sqrt{\frac{3 - \sqrt{5}}{5 - \sqrt{5}}} \quad ;$$

And

$$|\langle \bar{s}_0, \bar{s}_{11} \rangle| = |\langle \bar{s}_1, \bar{s}_9 \rangle| = |\langle \bar{s}_2, \bar{s}_{10} \rangle| = |\langle \bar{s}_3, \bar{s}_6 \rangle| = |\langle \bar{s}_4, \bar{s}_7 \rangle| =$$

$$|\langle \bar{s}_5, \bar{s}_8 \rangle| = 0.$$

Using the signal sets just presented, we substitute in hyperplane equation (Eq.(108)) and the equation for the boundary of the caps (Eq.(109)). We now graphically present the results for $M = 2, 3, 4, 6$, and 12, showing the relationship of the hyperplanes and the caps both in the u, χ coordinate system and in the x, y, z coordinate system.

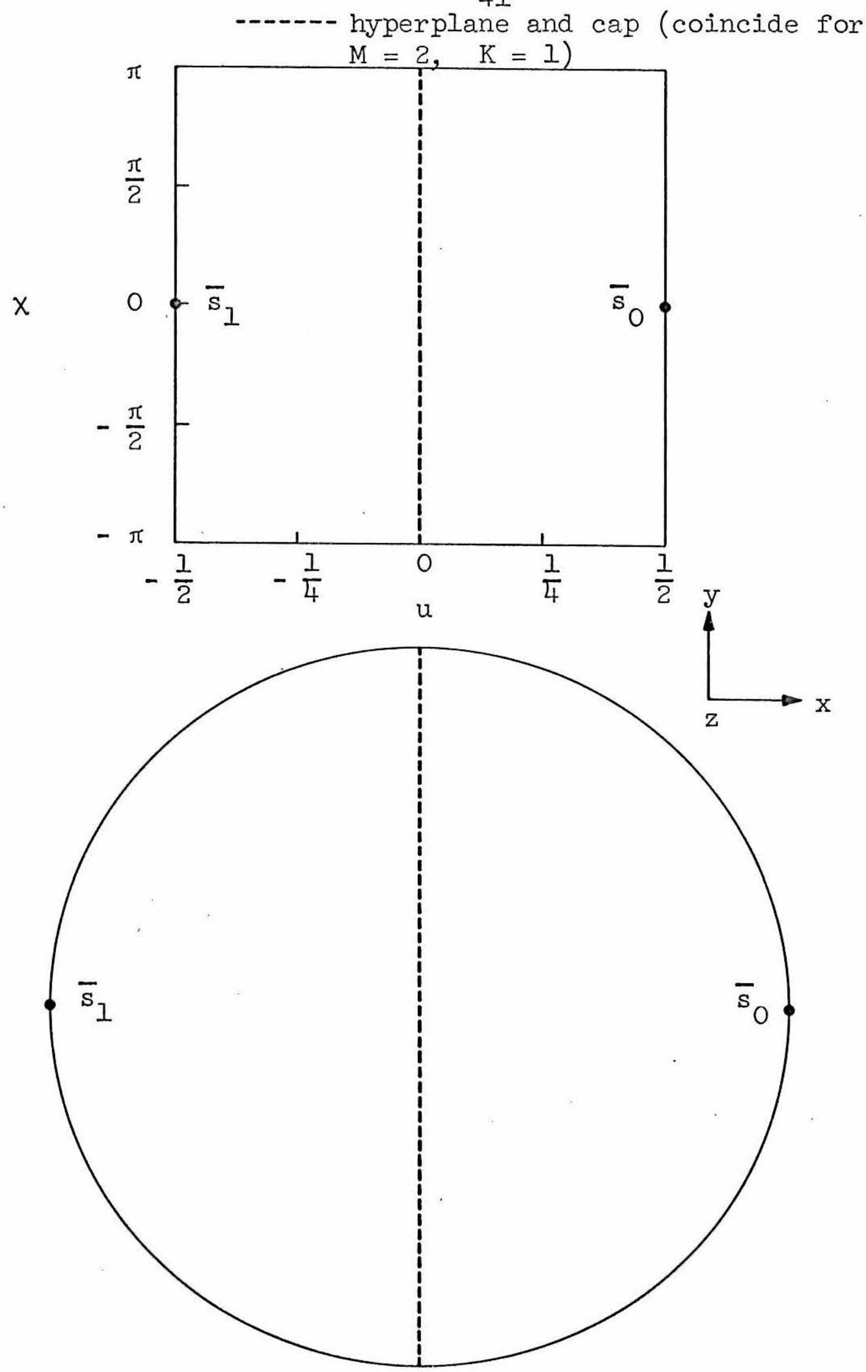

FIGURE VII-1

The optimal signal set for $M = 2$, $K = 1$ shown in the u, χ and the x, y, z co-ordinate system.

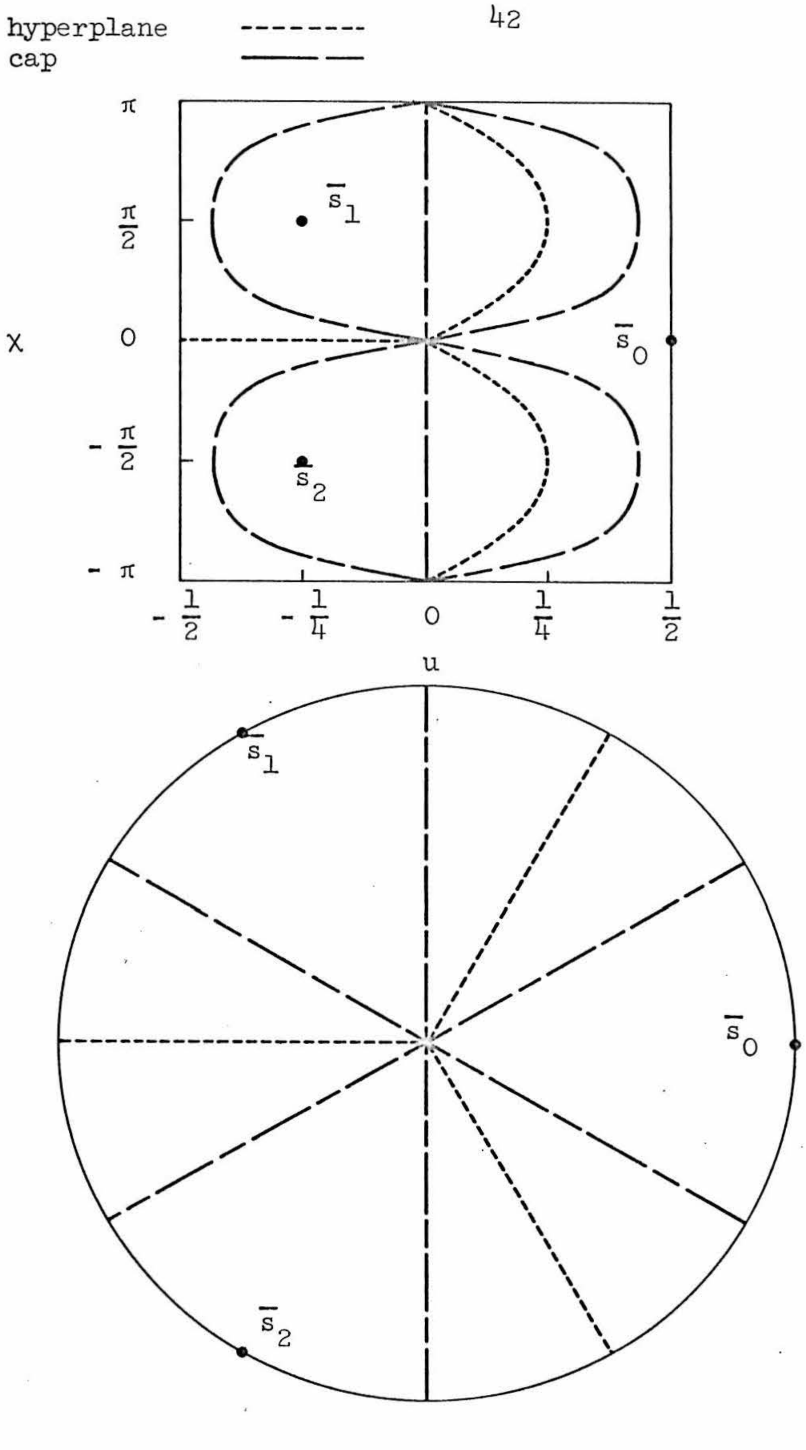

FIGURE VII-2

The optimal signal set for $M = 3$, $K = 3$ shown in the u, χ and the x, y, z co-ordinate system.

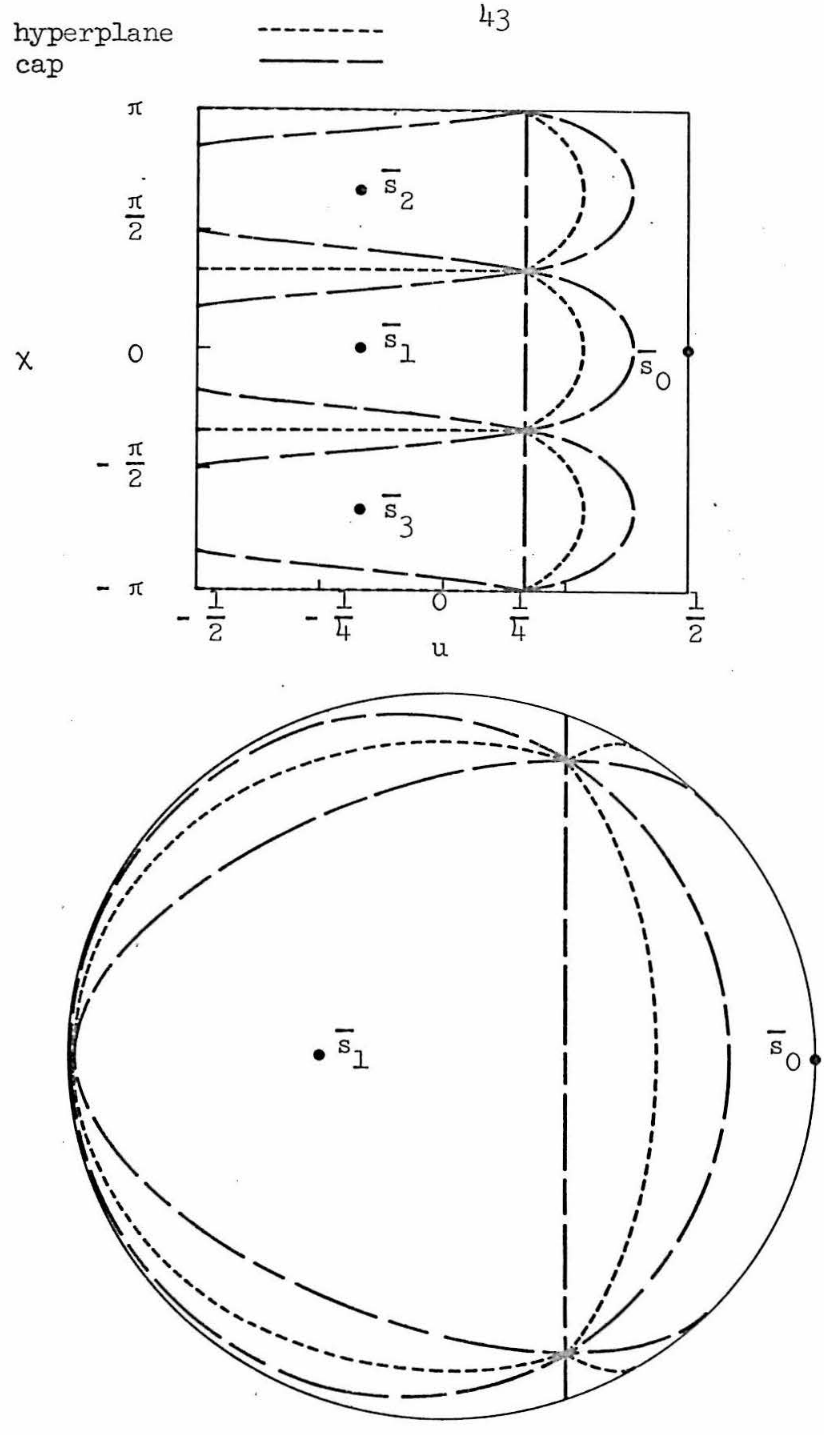

FIGURE VII-3

The optimal signal set for M = 4, K = 6 shown in the u, χ and the x, y, z co-ordinate system.

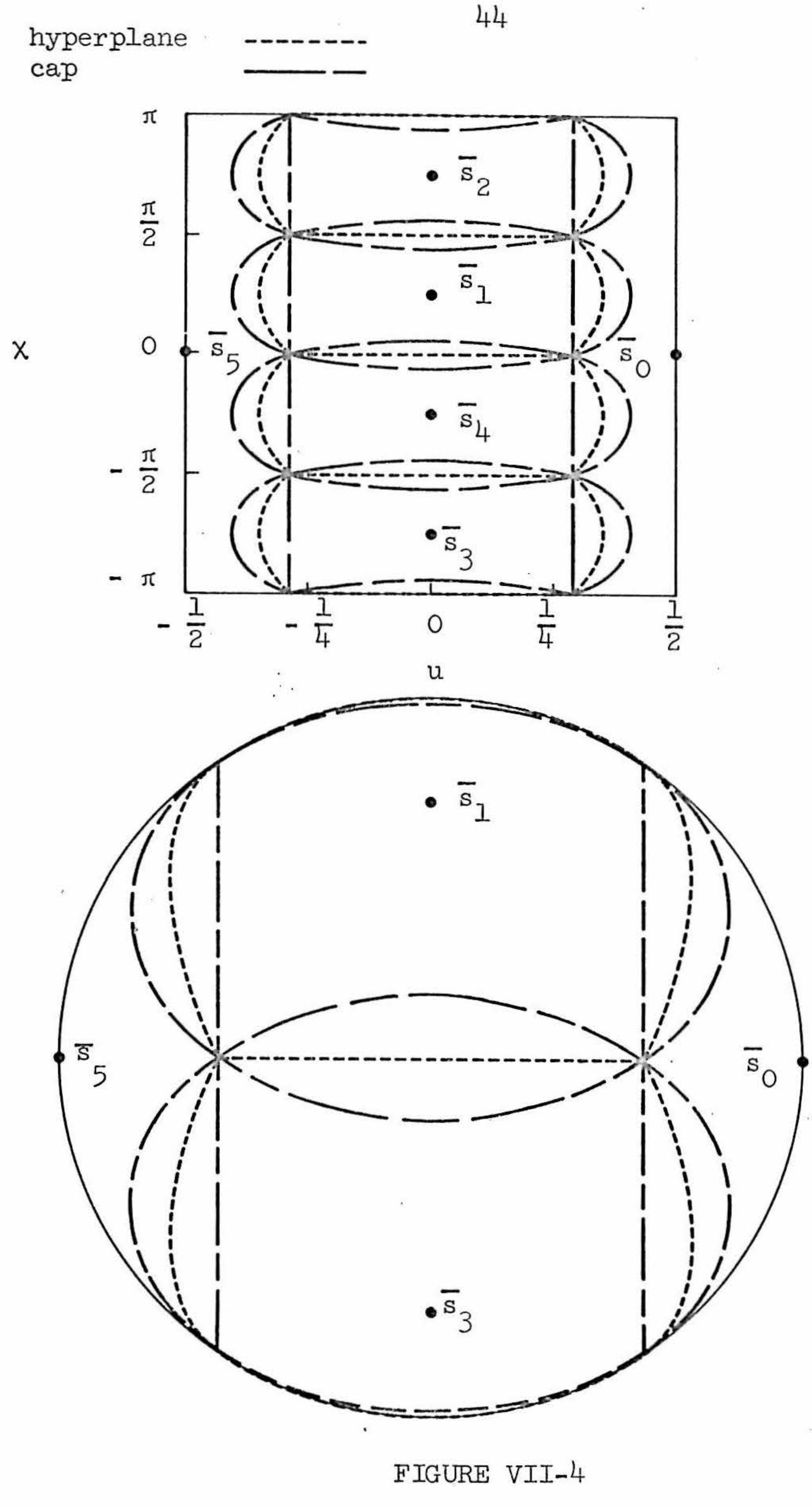

FIGURE VII-4

The optimal signal set for $M = 6$, $K = 12$ shown in the u, χ and the x, y, z co-ordinate system.

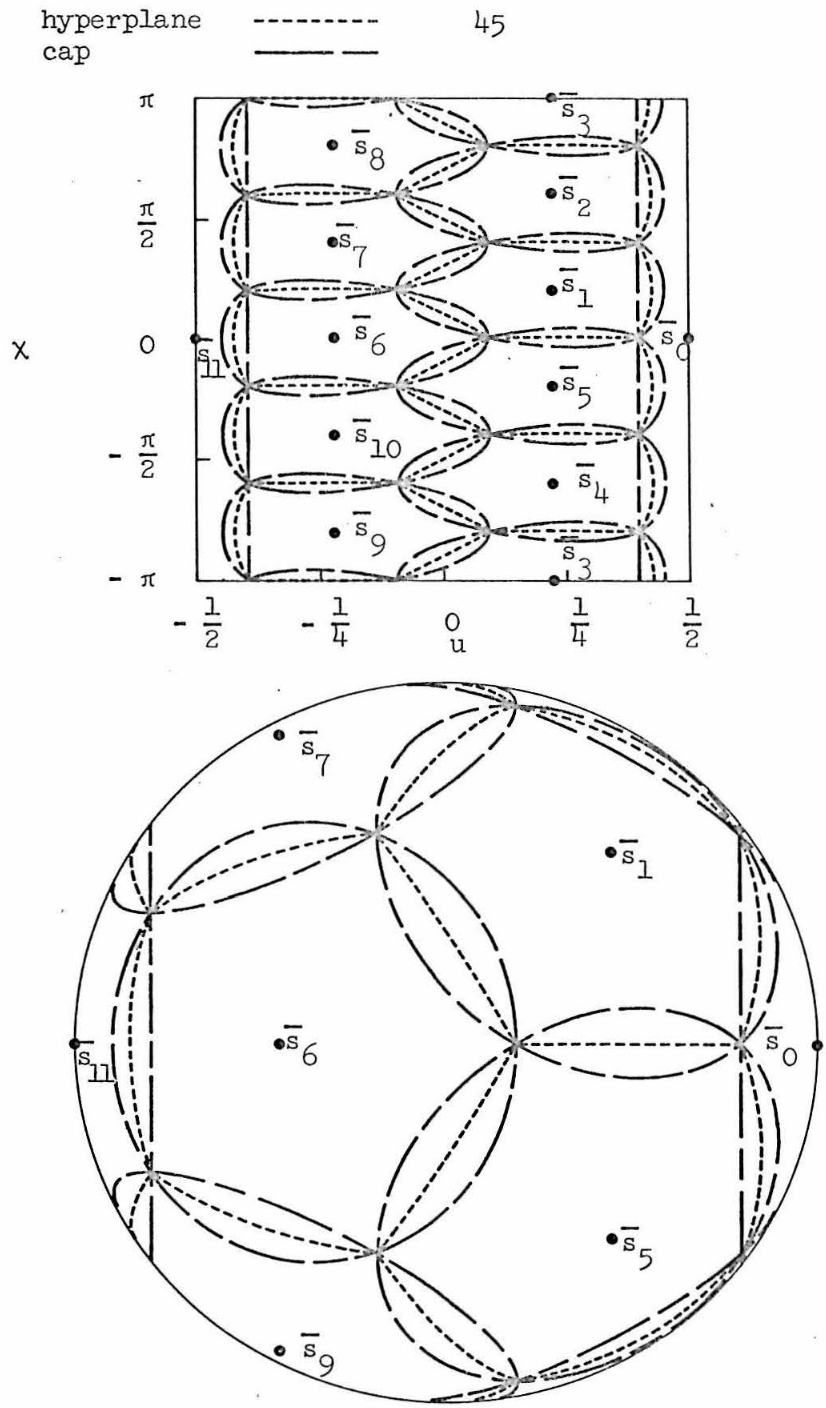

FIGURE VII-5

The optimal signal set for $M = 12$, $K = 30$ shown in the u, χ and the x, y, z co-ordinate system.

CHAPTER VIII

CALCULATION OF THE PROBABILITY OF ERROR FOR PHASE-INCOHERENT ORTHOGONAL SIGNALS

Using the notation of Chapter I and assuming equal energy and equi-probable signals, we have

$$P(\bar{r}|\bar{s}_i) = \frac{1}{\pi^N} e^{-\|\bar{r}-A\bar{s}_i e^{j\theta}\|^2} \qquad (118)$$

The probability of correct decoding assuming orthogonal signals is then

$$P[\|\bar{r}-A\bar{s}_i e^{j\theta}\|^2 < \min_{k\neq i}\|\bar{r}-A\bar{s}_k e^{j\theta}\|^2]$$

$$= P[|\langle\bar{r},\bar{s}_i\rangle| > \max_{k\neq i} |\langle\bar{r},\bar{s}_k\rangle|]$$

$$= P[|\langle A\bar{s}_i e^{j\theta} + \bar{n},\bar{s}_i\rangle|^2 > \max_{k\neq i} |\langle A\bar{s}_k e^{j\theta} + \bar{n},\bar{s}_k\rangle|^2]$$

$$= P[|Ae^{j\theta} + \langle\bar{n},\bar{s}_i\rangle|^2 > \max_{k\neq i} |\langle\bar{n},\bar{s}_k\rangle|^2] \qquad (119)$$

Since we are using orthogonal signals, let the orthonormal basis functions be $\varphi_i(t) = s_i(t)$. Then the probability of correct decoding can be written as

$$P[|Ae^{j\theta} + n^i|^2 \geq \max_{k\neq i} |n^k|^2] \qquad (120)$$

where the n^i are independent complex normal distributed with zero mean and variance one. Since the statistics of the noise do not depend upon which signal was transmitted, we have that the probability of being correct is independent of which signal was transmitted.

$$P_C = E[P(|Ae^{j\theta} + n^i|^2 > \max_{k \neq i} |n^k|^2 \quad |n^i)]$$

$$= E[\prod_{k \neq i} P(|Ae^{j\theta} + n^i|^2 > |n^k|^2 |n^i)] \tag{121}$$

Now, $|n^k|^2$ is equal to the sum of the squares of its real and imaginary parts, and these parts are each identically distributed independent, normal, mean zero, and variance one half. Therefore, $|n^k|^2$ has a chi-squared density of rank two and mean one.

$$P(|n^k|^2) = \begin{cases} e^{-|n^k|^2} & |n^k|^2 > 0 \\ 0 & \text{elsewhere} \end{cases} \tag{122}$$

$$P[|Ae^{j\theta} + n_i|^2 > |n_k|^2] = \int_0^{|Ae^{j\theta} + n^i|^2} e^{-|n^k|^2} \, d|n^k|^2$$

$$= 1 - e^{-|Ae^{j\theta} + n^i|^2} \tag{123}$$

Resulting in the probability of correct being

$$P_c = E\left[\left(1-e^{-|Ae^{j\theta} + n^i|^2}\right)^{N-1}\right] \tag{124}$$

Let $n = n^i e^{-j\theta}$ then n and n^i are identically distributed random variables since n^i is invariant under phase translations. Therefore, we obtain

$$P_c = E\left[\left(1-e^{-(A+n)}\right)^{N-1}\right] \tag{125}$$

By expanding in a binomial series, we obtain

$$P_c = \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \, E\left[e^{-k|A+n|^2}\right] \tag{126}$$

Now, since n is a complex normal random variable with zero mean and variance one, we have

$$E\left[e^{-k|A+n|^2}\right] = \frac{1}{\pi} \int_n e^{-[(n)^2 + k(A+n)^2]} dn$$

$$= \frac{1}{k+1} e^{-\frac{kA^2}{k+1}} \left[\int_n \frac{k+1}{\pi} e^{-(k+1)\left[n + \frac{kA}{k+1}\right]^2} dn \right] \tag{127}$$

We see that the above integral in brackets is the integral of a complex normal density of mean $\frac{kA}{k+1}$ and variance $\frac{1}{k+1}$ and is therefore equal to unity. We therefore have that the probability of being correct is

$$P_c = \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} \frac{e^{-\frac{kA^2}{k+1}}}{k+1} \tag{128}$$

The probability of error is $1 - P_c$ and may be written as

$$P_e = \frac{1}{N} e^{-A^2} \sum_{k=2}^{N} (-1)^k \binom{N}{k} e^{A^2/k} \tag{129}$$

CHAPTER IX

COMPARISON OF THE PROBABILITY OF ERROR BETWEEN THE GLOBALLY OPTIMAL SIGNALS IN C^2 AND ORTHOGONAL SIGNALS

In the previous Chapter we obtained the probability of correct decoding for orthogonal phase-incoherent signals. In this section, we obtain the probability of correct decoding for the globally optimal phase-incoherent signal sets in complex two space. We then calculate and present the probability of error for both these cases as a function of signal-to-noise ratio.

By using Eqs.(20), (52), and (67) we may obtain the probability of correct decoding for unit of signal-to-noise ratio (A = 1).

$$P_c = \int_0^\infty \frac{1}{\pi^2} r^3 e^{-(r^2+1)} U(r)\, dr \tag{130}$$

where $$U(r) = 2\pi^2 \int_v^{1/2} P_r(\sqrt{u+\tfrac{1}{2}})\, du - \frac{2K}{M} \int_v^{\tau} k_t(u)\, P_r(\sqrt{u+\tfrac{1}{2}})\, du \tag{131}$$

but $$P_r(\sqrt{u+\tfrac{1}{2}}) = I_0(2r\sqrt{u+\tfrac{1}{2}}) \tag{132}$$

$$\therefore P_c = 2\int_0^\infty r^3 e^{-(r^2+1)} \left\{ \int_v^{1/2} I_0(2r\sqrt{u+\tfrac{1}{2}})\, du - \frac{1}{2\pi x}\int_v^{\tau} k_t(u) I_0(2r\sqrt{u+\tfrac{1}{2}})\, du \right\} dr \tag{133}$$

In order to calculate the probability of being correct for any signal-to-noise ratio, we see from Eq.(12), Chapter I, that the above equation must be modified to become

$$P_c = 2\int_0^\infty r^3 e^{-(r^2+A^2)} \left\{ \int_v^{1/2} I_0(2Ar\sqrt{u+\tfrac{1}{2}})\, du - \frac{1}{2\pi x}\int_v^{\tau} k_t(u) I_0(2Ar\sqrt{u+\tfrac{1}{2}})\, du \right\} dr \tag{134}$$

where

$$k_t(u) = 2\pi \text{ arcos } \frac{u}{t\sqrt{1/4 - u^2}} \tag{135}$$

and

$$\tau = \frac{t}{2\sqrt{1 + t^2}} \tag{136}$$

The probability of error can now be calculated from the above equation by $P_e = 1 - P_c$ for the signal-to-noise ratios desired. Figure IX-1 shows a performance comparison of the orthogonal signal $(M = N)$ and the globally optimal signals in C^2 as a function of signal-to-noise ratio A^2. Figure IX-2 shows the degradation in probability of error as the number of signals increases for fixed signal-to-noise ratios.

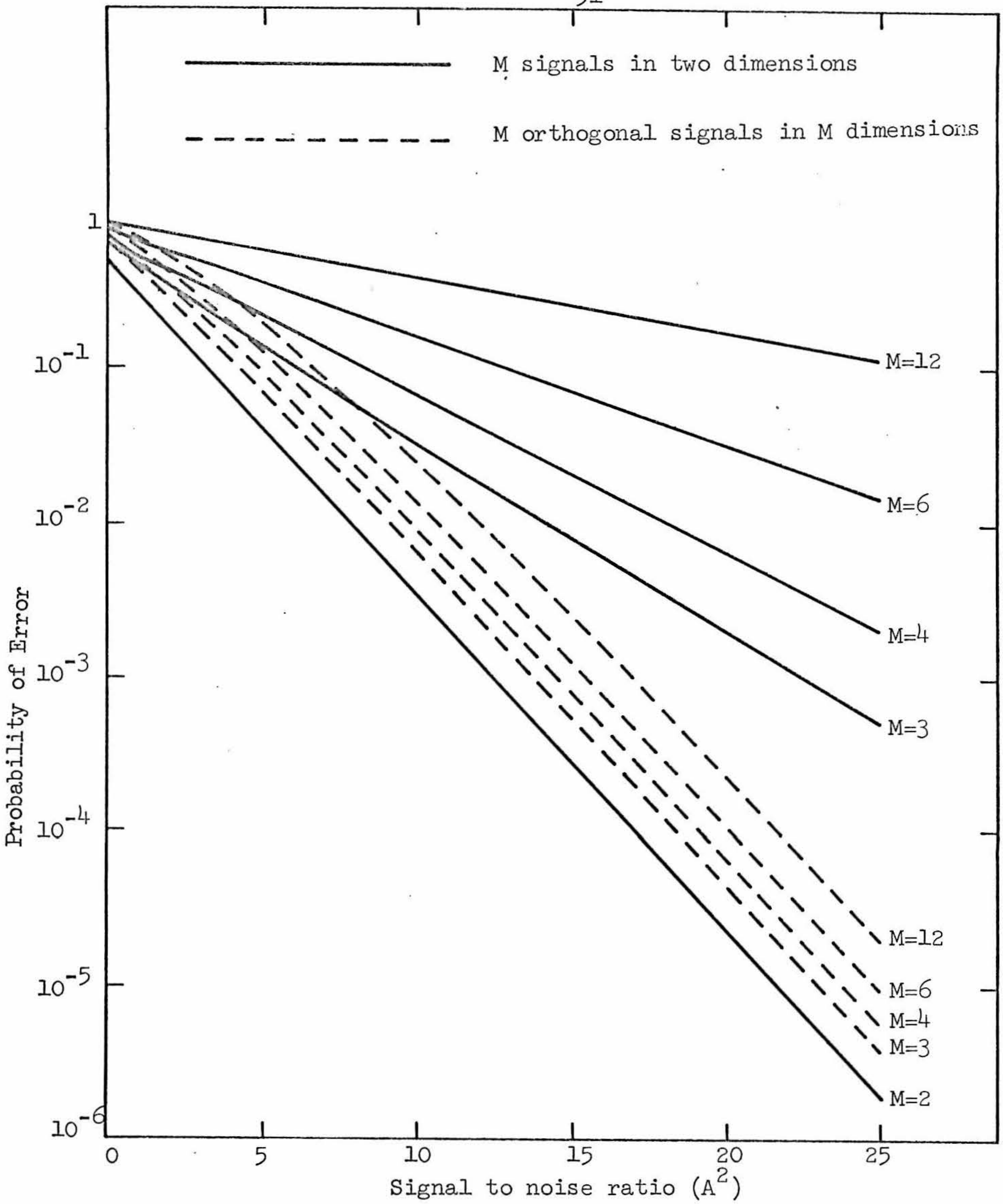

FIGURE IX-1

Comparison of the probability of error for orthogonal signals and globally optimum signals in C^2.

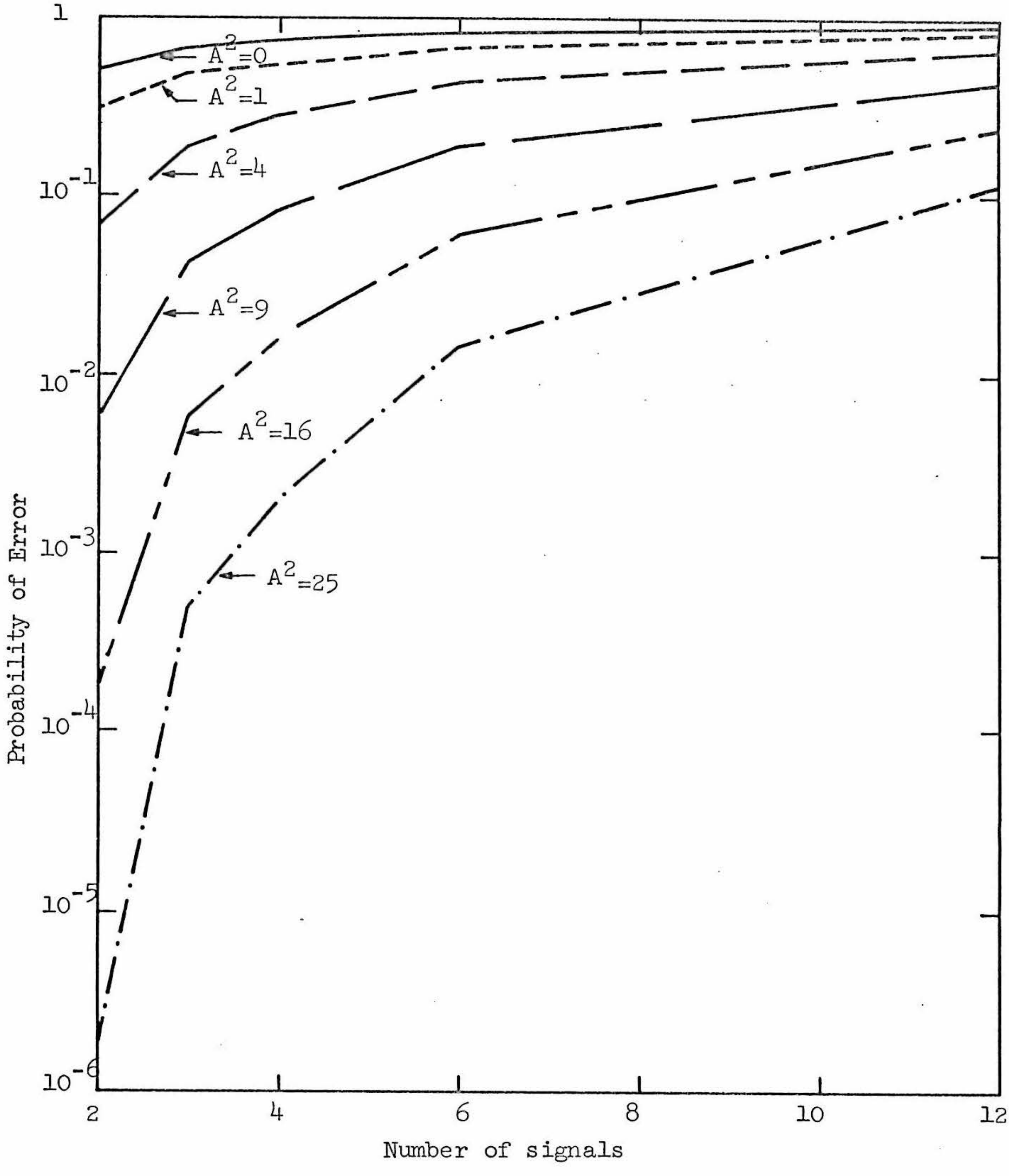

FIGURE IX-2

Comparison of the probability of error for globally optimum signals in C^2 as a function of the number of signals for fixed signal to noise ratio (A^2).

CHAPTER X

CONCLUSIONS

We have formulated a set of conditions for the global optimality of M equally probable, equal energy phase-incoherent signals in N complex dimensions. In this method, we consider the signal vector as points on the unit sphere in C^2; and, by means of a geometric argument similar to that of Landau and Slepian, we proved the validity of these conditions for $N = 2$.

We then perform the unit sphere in C^2 onto the unit sphere in three real-dimensional Euclidean space. Using this transformation, we map the hyperplane equation and the equation for the boundary of the cap. We then establish that the only signal sets in C^2 that can be shown to be globally optimal by this method are $M = 2, 3, 4, 6,$ and 12. We determine that for the globally optimal signal sets there are respectively 1, 2, 3, 4, and 5 hyperplanes determining the optimum decision regions. Next, we determine what these globally optimal signal sets are for these values of M's. These sets are those for which the inner products between a given signal vector and the ones making up the decision region about the given vector are $\langle \bar{s}_i, \bar{s}_j \rangle = 0$ for $M = 2$, $|\langle \bar{s}_i, \bar{s}_j \rangle| = \frac{1}{2}$ for $M = 3$, $|\langle \bar{s}_i, \bar{s}_j \rangle| = \frac{1}{\sqrt{3}}$ for $M = 4$, $|\langle \bar{s}_i, \bar{s}_j \rangle| = \frac{1}{\sqrt{2}}$ for $M = 6$, and $|\langle \bar{s}_i, \bar{s}_j \rangle| = \sqrt{\frac{2}{5 - \sqrt{5}}}$ for $M = 12$.

We then compare the probability of error performance between the globally optimal signals in C^2 and the orthogonal signal sets in C^M as a function of the signal-to-noise ratio.

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